Lecture 35: Absolute convergence + the Root & Ratio tests

Let's start with sequences/series of positive terms once more: assume $b_n > 0$ for all n.

Ratio test: Suppose $\rho := \lim_{n\to\infty} \frac{b_{n+1}}{b_n}$ exists. Then

- $\rho < 1 \implies \sum b_n$ converges
- $\rho > 1 \implies \sum b_n$ diverges.

Note: "$\rho < 1$" includes 0, and "$\rho > 1$" includes ∞.

Root test: Suppose $\mathcal{R} := \lim_{n\to\infty} \sqrt[n]{b_n}$ exists. Then

- $\mathcal{R} < 1 \implies \sum b_n$ converges
- $\mathcal{R} > 1 \implies \sum b_n$ diverges.

Note: For a geometric series $\sum r^n$, we get $\rho = r = \mathcal{R}$.

Here is why the ratio test works:

- if $\rho < 1$, take $r \in (\rho, 1)$, and N large enough that $n \geq N \implies \frac{b_{n+1}}{b_n} < r$. Then

$$b_{N+k} < r\, b_{N+k-1} < r^2 b_{N+k-2} < \cdots < r^k b_N$$

and

$$\sum_{n=N}^{\infty} b_n \leq \underbrace{\sum_{n=N}^{\infty} b_N r^{n-N}}_{\text{think: } n = N+k} = \frac{b_N}{r^N} \sum_{n=N}^{\infty} r^n$$

which converges (as $r<1$).

By the Basic Comparison (balloon) test, $\sum b_n$ converges. $\square$

- if $\rho > 1$, there is N such that $n \geq N \Rightarrow \frac{b_{n+1}}{b_n} > 1$.
Then $b_n > b_{n-1} > \cdots > b_N$ for any $n > N$, and so $\lim_{n\to\infty} b_n \neq 0$. By the Test for Divergence, $\sum b_n$ diverges. $\square$

- <u>Note</u>: if b_n is any rational function of n $\left(\text{i.e. } \frac{\text{polynomial}}{\text{polynomial}}\right)$, ρ (and r) $= 1$. For instance

$\rightarrow$ $b_n = \frac{1}{n} \Rightarrow \rho = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1$

$\rightarrow$ $b_n = \frac{1}{n^2} \Rightarrow \rho = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$

but $\sum \frac{1}{n}$ diverges whereas $\sum \frac{1}{n^2}$ converges. So the message is (A) the tests are inconclusive if ρ or $r = 1$ and (B) you shouldn't use them on rational b_n's because they'll always be inconclusive. (Use limit comparison with p-series on such series.)

- What about the root test? It turns out that
  - $\rightarrow$ If both ρ and r exist, they are <u>equal</u>
  - $\rightarrow$ If ρ exists, then so does r

So in some sense they're the same test. $\leftarrow$ <u>Upshot</u>: use whichever is more convenient!

[As we saw above, if ρ exists and ϵ is small, there are constants c and C such that for n large enough

$$c(\rho-\epsilon)^n < b_n < C(\rho+\epsilon)^n$$

$\leftarrow$ we called this "r"

take $\sqrt[n]{\ }$

$$c^{1/n}(\rho-\epsilon) < \sqrt[n]{b_n} < C^{1/n}(\rho+\epsilon)$$

$\lim_{n\to\infty}$

$$\rho - \epsilon < r < \rho + \epsilon$$

and by taking $\epsilon \to 0$ we see $\rho = R$.]

Ex 1/ $\sum \underbrace{\frac{n!}{n^n}}_{b_n}$ has $\rho = \lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \frac{\cancel{(n+1)!}^{\,n+1}\ n^n}{(n+1)^{n+1}\ \cancel{n!}} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n}$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$$

Converges! $\Leftarrow$ //

Ex 2/ $\sum \underbrace{\frac{(n!)^2}{(2n)!}}_{b_n}$ has $\rho = \lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \frac{((n+1)!)^2\ (2n)!}{(2n+2)!\ (n!)^2}$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1$$

Converges! $\Leftarrow$ //

Ex 3/ $\sum \left(\frac{n}{3n+2}\right)^n$ has $R = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{3n+2}\right)^n} = \lim_{n\to\infty} \frac{n}{3n+2} = \frac{1}{3}$

$\Rightarrow$ Converges! //

Does a series like

$$\sum a_n = 1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \cdots$$

$+ , + , - , + , + , -$ ($\rightsquigarrow$ not "alternating")

converge or diverge? The associated series

$$\sum |a_n| = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots = \sum \frac{1}{n^2}$$

converges, so can we make use of that? YES!

<u>Absolute Convergence Test</u>: If $\sum |a_n|$ converges, so does $\sum a_n$.

<u>Why it works</u>: let $c_n = a_n + |a_n|$, so $a_n = c_n - |a_n|$, and $0 \le c_n \le 2|a_n|$. Basic C.T. + convergence of $\sum |a_n|$ $\Rightarrow \sum c_n$ converges $\Rightarrow \sum a_n = \sum c_n - \sum |a_n|$ converges. $\square$

Definition: (i) A series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges. [stronger than just convergence!]

(ii) A series $\sum a_n$ which is convergent, but NOT absolutely convergent, is called conditionally convergent.

To establish AC (= absolute convergence) for $\sum a_n$, we apply the root &/or ratio tests to $\sum |a_n|$.

Ex 4/ $\sum \underbrace{(-1)^{n+1} \frac{3^n}{n!}}_{a_n}$ $\Rightarrow$ $|a_n| = \frac{3^n}{n!}$, and

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim \frac{3^{\cancel{n+1}}\,\cancel{n!}}{\underbrace{\cancel{(n+1)!}}_{n+1}\,\cancel{3^n}} = \lim_{n\to\infty} \frac{3}{n+1} = 0.$$

$\Rightarrow \sum a_n$ is AC. //

Ex 5/ $\sum \underbrace{\frac{\cos(n!)}{n^2}}_{a_n}$ $\rightsquigarrow$ $\sum |a_n| = \sum \frac{|\cos(n!)|}{n^2} \le \sum \frac{1}{n^2}$ converges

$\underset{\text{Basic C.T.}}{\Longrightarrow} \sum |a_n|$ convergent

$\Longrightarrow \sum a_n$ AC. //

Ex 6/ $\sum \underbrace{(-1)^{n+1} \frac{1}{\sqrt{n}}}_{a_n}$ Convergent by Alternating Series test

$\sum |a_n| = \sum \frac{1}{n^{1/2}}$ divergent by p-series test

$\Rightarrow \sum a_n$ is conditionally convergent. //

A couple of weird examples:

Ex 7/ $1 + \frac{1}{4} + \frac{1}{2} + \frac{1}{16} + \frac{1}{8} + \frac{1}{64} + \frac{1}{32} + \frac{1}{256} + \frac{1}{128} + \cdots$

is a "rearrangement" of the series $\sum \frac{1}{2^n}$. For this rearrangement, ρ doesn't exist: $\frac{a_{n+1}}{a_n}$ alternates between 2 and $\frac{1}{8}$. But r exists and = geometric mean of those two: $\frac{1}{2}$.
The series converges and the rearrangement doesn't affect its sum //

Ex 8/ $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + - \cdots = \ln(2)$

is <u>conditionally</u> convergent. Rearranging <u>can</u> change the sum (= limit of partial sums S_n). e.g.

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots =$$

$$1 + 0 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + 0 + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + 0 + \cdots =$$

$$\left(0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + 0 - \frac{1}{10} + \cdots\right)$$

$$+ \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots\right) =$$

both converge

$$\frac{1}{2} \sum (-1)^{n+1} \frac{1}{n} + \sum (-1)^{n+1} \frac{1}{n} = \frac{1}{2} \ln 2 + \ln 2$$

$$= \frac{3}{2} \ln 2.$$

This isn't that surprising since the sum of the "odd" $\frac{1}{n}$'s on their own is ∞, and same for the "even" $\frac{1}{n}$'s. //