Lecture 35: Absolute convergence + the Root & Ratio tests

Let's start with sequences of positive terms once more:

Assume $b_n > 0$ for all $n$.

**Ratio test:** Suppose $\rho := \lim_{n \to \infty} \frac{b_{n+1}}{b_n}$ exists. Then

- $\rho < 1 \implies \sum b_n$ converges.
- $\rho > 1 \implies \sum b_n$ diverges.

(\textit{Note:} "\(\rho < 1\)" includes 0, and "\(\rho > 1\)" includes \(\infty\).

**Root test:** Suppose $\rho := \lim_{n \to \infty} \sqrt[n]{b_n}$ exists. Then

- $\rho < 1 \implies \sum b_n$ converges.
- $\rho > 1 \implies \sum b_n$ diverges.

(\textit{Note:} For a geometric series $\sum r^n$, we get $\rho = r = r_e$.

Here is why the ratio test works:

- if $\rho < 1$, take $r \in (\rho, 1)$, and $N$ large enough that
  $n \geq N \implies \frac{b_{n+1}}{b_n} < r$. Then

  $b_{N+1} < rb_{N+2} < r^2 b_{N+3} < \ldots < r^k b_N$,

  and $\sum_{n=N}^{\infty} b_n \leq \sum_{n=N}^{\infty} b_n r^{n-N} = \frac{b_N}{r^N} \sum_{n=N}^{\infty} r^n$ which converges
  (think $n = N + k$)

  By the Basic Comparison (balloon) test, $\sum b_n$ converges.
• if \( p > 1 \), there is \( N \) such that \( n \geq N \Rightarrow \frac{b_{n+1}}{b_n} > 1 \).
Then \( b_n > b_{n-1} > \ldots > b_N \) for any \( n > N \), and so
\[
\lim_{n \to \infty} b_n \neq 0. \text{ By the Test for Divergence, } \sum b_n \text{ diverges.} \]

- Note: if \( b_n \) is any rational function of \( n \) (i.e. \( \frac{\text{polynomial}}{\text{polynomial}} \)), \( p \) (and \( n \)) = 1. For instance

\[
\Rightarrow b_n = \frac{1}{n^n} \Rightarrow p = \lim_{n \to \infty} \frac{1}{n^{n+1}} = \lim_{n \to \infty} \frac{1}{n^{n+1}} = 1
\]

\[
\Rightarrow b_n = \frac{1}{n^2} \Rightarrow p = \lim_{n \to \infty} \frac{1}{(n+1)^2} = 1
\]

but \( \sum \frac{1}{n^n} \) diverges whereas \( \sum \frac{1}{n^2} \) converges. So the message is (A) the tests are inconclusive if \( p = 1 \) and (B) you shouldn't use them on rational \( b_n \)'s because they'll always be inconclusive. (Use limit comparison with \( p \)-series on such series.)

- What about the root test? It turns out that
  \[
  \Rightarrow \text{ If both } p \text{ and } R \text{ exist, they are equal}
  \]
  \[
  \Rightarrow \text{ If } p \text{ exists, then so does } R
  \]
So in some sense they're the same test. \( \sqrt{\text{Root test is more convenient!}} \)

[As we saw above, if \( p \) exists and \( \varepsilon \) is small, there are constants \( c \) and \( C \) such that for \( n \) large enough
\[
c(p-\varepsilon)^n < b_n < C(p+\varepsilon)^n
\]

\[
\Rightarrow \lim_{n \to \infty} \left( c(p-\varepsilon) \right)^{1/n} < \sqrt[n]{b_n} < C^{1/n}(p+\varepsilon)
\]

\[
\Rightarrow p-\varepsilon < R < p+\varepsilon
\]

and by taking \( \varepsilon \to 0 \) we see \( p = R \).]
Ex 1 \[ \sum \frac{n!}{n^n} \] has \( p = \lim_{n \to \infty} \frac{b_n}{b_{n-1}} = \lim_{n \to \infty} \frac{n+1}{(n+1)^{n+1}} = \lim_{n \to \infty} \frac{n^n}{(n+1)^n} \]

Converges! \( \leq \) \( \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n = e < 1 \)

Ex 2 \[ \sum \frac{(n!)^2}{(2n)!} \] has \( p = \lim_{n \to \infty} \frac{b_n}{b_{n-1}} = \lim_{n \to \infty} \frac{(n+1)!}{(2n+2)!} \frac{(2n)!}{(n)!^2} \]

Converges! \( \leq \) \( \lim_{n \to \infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1 \)

Ex 3 \[ \sum \frac{n}{3n+2} \] has \( R = \lim_{n \to \infty} \sqrt[n]{\frac{n}{3n+2}} = \lim_{n \to \infty} \frac{n}{3n+2} = \frac{1}{3} \)

\( \Rightarrow \) converges!

Does a series like

\[ \sum a_n = 1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \ldots \]

\( (+, +, -, +, +, - \ldots \text{not "alternating"}) \)

Converge or diverge? The associated series

\[ \sum |a_n| = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \ldots = \sum \frac{1}{n^2} \]

converges, so can we make use of that? YES!

**Absolute Convergence Test:** If \( \sum |a_n| \) converges, so does \( \sum a_n \).

Why it works: let \( c_n = a_n + |a_n| \), so \( a_n = c_n - |a_n| \),

and \( 0 \leq c_n \leq 2|a_n| \). Basic C.T. + Converge of \( \sum |a_n| \)

\( \Rightarrow \sum c_n \) converges \( \Rightarrow \sum a_n = \sum c_n - \sum |a_n| \) converges.
Definition: (i) A series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges. [Stronger than just convergence.]

(ii) A series $\sum a_n$ which is convergent, but NOT absolutely convergent, is called conditionally convergent.

To establish AC (= absolute convergence) for $\sum a_n$, we apply the root test to $\sum |a_n|$.

Ex 4/ $\sum \frac{(-1)^{n+1} 3^n}{n!} \Rightarrow \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{3^{n+1}}{n+1} = 0$.

$\Rightarrow \sum a_n$ is AC.

Ex 5/ $\sum \frac{\cos(n!)}{n^2} \Rightarrow \sum |a_n| = \sum \frac{|\cos(n!)|}{n^2} \leq \sum \frac{1}{n^2}$ Converges.

$\Rightarrow \sum |a_n|$ convergent.

Basic C.T.

$\Rightarrow \sum a_n$ AC.

Ex 6/ $\sum \frac{(-1)^{n+1}}{\sqrt{n}}$ convergent by Alternating Series test.

$\sum |a_n| = \sum \frac{1}{\sqrt{n}}$ divergent by p-series test.

$\Rightarrow \sum a_n$ is conditionally convergent.
A couple of weird examples:

Ex 7 / 1 + \frac{1}{4} + \frac{1}{2} + \frac{1}{6} + \frac{1}{8} + \frac{1}{64} + \frac{1}{32} + \frac{1}{256} + \frac{1}{128} + \ldots

is a "rearrangement" of the series \( \sum \frac{1}{2^n} \). For this rearrangement, \( p \) doesn't exist: \( \lim \frac{a_n}{a_{n+1}} \), alternates between 2 and \( \frac{1}{8} \). But \( R \) exists and \( \sqrt[4]{2} \) is the geometric mean of these two, \( \frac{1}{2} \).

The series converges and the rearrangement doesn't affect its sum.

Ex 8 / 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \ldots = \ln(2)

is conditionally convergent. Rearranging can change the sum (= limit of partial sums \( S_n \)).

\[
1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \ldots =
\]

\[
1 + 0 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + 0 + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + 0 + \ldots =
\]

\[
\left(1 + \frac{1}{2} + 0 - \frac{1}{3} + 0 + \frac{1}{4} + 0 - \frac{1}{5} + 0 - \frac{1}{6} + 0 - \frac{1}{7} + 0 - \frac{1}{8} + 0 - \frac{1}{9} + 0 - \frac{1}{10} + \ldots\right) + \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \ldots\right) =
\]

\[
\frac{1}{2} \sum (-1)^{n+1} \frac{1}{n} + \sum (-1)^{n+1} \frac{1}{n} = \frac{1}{2} \ln 2 + \ln 2
\]

\[
= \frac{3}{2} \ln 2.
\]

This isn't that surprising since the sum of the "odd" \( \frac{1}{n} \)'s on their own is \( \frac{3}{2} \ln 2 \), and same for the "even" \( \frac{1}{n} \)'s.