Lecture 36 : Power series

So far we have been studying series of real numbers.

We now turn to series of functions, that is, "$\sum a_n(x)$" rather than "$\sum a_n$". A priori these could be any kind of functions we like — $\sum c_n e^{nx}$, $\sum c_n \sin(nx)$, etc. — but we will consider exclusively series of powers of x $\sum_{n=0}^{\infty} c_n x^n$, known as power series.

or more generally, of $(x-a)$

For any series of functions, there are two important questions to ask:

Ⓐ For which x does the series converge?

Ⓑ To what function $S(x)$ does it sum (where it converges)?

Ex 1/ $\sum_{n=0}^{\infty} a x^n$ converges for $x \in (-1,1)$, and has sum $S(x) = \frac{a}{1-x}$ there. //

(A) (B)

We'll focus on Ⓐ in today's lecture, i.e. on determining the convergence set of a power series $\sum \underbrace{c_n (x-a)^n}_{a_n}$.

n^{th} coefficient

Ex 2/ $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$

Apply the (absolute) ratio test:

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x|^{n+1}\, n!}{(n+1)!\,|x|^n} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0,$$

regardless of x. So the series is absolutely convergent for all $x \in \mathbb{R}$. //

Ex 3/ $\sum_{n=0}^{\infty} n^n x^n$ Apply the (absolute) root test:

$$\pi = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} n|x| = \infty \quad \text{unless } x = 0.$$

So the series converges only for $x = 0$. //

Ex 4/ $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)2^n}$ Apply the ratio test again:

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x|^{n+1}(n+1)2^n}{(n+2)2^{n+1}|x|^n} = \lim_{n\to\infty} \frac{n+1}{n+2} \cdot \lim_{n\to\infty} \frac{|x|}{2}$$

$$= \frac{|x|}{2} \quad \text{is } < 1 \text{ for } x \in (-2,2) \Rightarrow AC \text{ absolute convergence}$$

$$> 1 \text{ for } x \in (-\infty,-2) \cup (2,\infty) \Rightarrow D \text{ divergence}$$

Test is inconclusive for $x = \pm 2$.

- $x = 2$: Series becomes $\sum_{n=0}^{\infty} \frac{1}{n+1} \longrightarrow D$ (by limit comp. to harmonic series)
- $x = -2$: becomes $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \longrightarrow C$ (by alternating series test)

So convergence set is $[-2, 2)$. //

Ex 5/ $\sum_{n=0}^{\infty} \frac{(x-1)^n}{(n+1)^2}$ $\quad \rho = \lim_{n\to\infty} \frac{|x-1|^{n+1}(n+1)^2}{(n+2)^2\,|x-1|^n}$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(n+2)^2} \cdot |x-1| = |x-1|.$$

By ratio test, we get AC for $x-1 \in (-1,1) \Leftrightarrow x \in (0,2)$, and divergence for $x<0$ or $x>2$. At $x = 2$ and 0, both $\sum \frac{1}{(n+1)^2}$ and $\sum \frac{(-1)^n}{(n+1)^2}$ converge. So conv. set is $[0,2]$. //

$\boxed{\text{Theorem}}$ The convergence set of a power series

$$\sum c_n (x-a)^n$$

is always an interval of one of the following 3 types:

(i) the single point $x = a$

(ii) an interval $(a-R, a+R)$, $[a-R, a+R)$, $(a-R, a+R]$, or $[a-R, a+R]$ — R is called the radius of convergence.

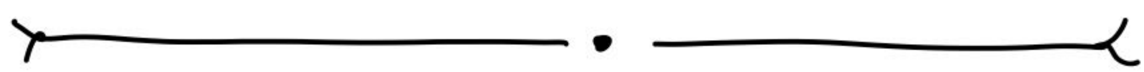
(iii) the entire real line $(-\infty, \infty) = \mathbb{R}$

Moreover, a power series always converges absolutely on the interior of its convergence set.

Though you've just seen examples of all 3 possibilities, how do we know this is always true? (Assume $a = 0$.)

If the series converges at $x_0 (\neq 0)$, then $\lim_{n\to\infty} c_n x_0^n = 0$, so $|c_n x_0^n| < 1$ for $n \geq N$. Now if $|x_1| < |x_0|$ then $|c_n x_1|^n = |c_n x_0^n| \left|\frac{x_1}{x_0}\right|^n < \left|\frac{x_1}{x_0}\right|^n$, and $\sum \left|\frac{x_1}{x_0}\right|^n$ conv. $\Rightarrow$ series converges [absolutely!] at x_1 by the Basic C.T.

This forces the convergence set to be a single interval, symmetric (except possibly for endpoints) about 0.



So given a series, how do you know which test to apply?

$\boxed{1}$ Is it a geometric series, or p-series? (Then done.)

$\boxed{2}$ Does it have $\lim_{n\to\infty} a_n \neq 0$? (If so, $\underline{\underline{D}}$.)

$\boxed{3}$ Is it a positive term series? If so, use one of:

- $\boxed{a}$ — Basic C.T.
- $\boxed{b}$ — Limit C.T. (esp. if $a_n = \frac{P(n)}{Q(n)}$ ← polynomials)
- $\boxed{c}$ — Ratio test (esp. if there is an $n!$, or a c^n)
- $\boxed{d}$ — Root test (esp. if have n^n or $P(n)^n$)
- $\boxed{e}$ — Integral test (if $a_n = f(n) \to 0$ with $\int f\,dx$ integrable)

$\boxed{4}$ Does it have both positive and negative terms?

- $\boxed{a}$ — If $a_n = (-1)^n b_n$, with b_n decreasing to 0, then $\underline{\underline{C}}$.
- $\boxed{b}$ — otherwise, use $\boxed{3}$ applied to $|a_n|$

$\boxed{5}$ Is it a power series?

- Use ratio/root test to find the convergence interval
- then use $\boxed{3}$ & $\boxed{4}$ to check for convergence at the endpoints of the interval.

$\boxed{\text{TRY}}$ $\sum_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^n$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{10^n}$, $\sum_{n=1}^{\infty} \frac{n!}{2^{n^2}}$, $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$

use: $\boxed{1}$ or $\boxed{3}$ d/e, $\boxed{1}$ or $\boxed{4}$ a, $\boxed{3}$ c, $\boxed{3}$ b

$\boxed{\text{TRY}}$ $\sum_{n=1}^{\infty} \frac{n x^n}{3^n}$ conv. set $= (-3, 3)$

$\sum_{n=2}^{\infty} \frac{(x+2)^n \ln(n)}{n \cdot 3^n}$ conv. set $= [-5, 1)$

As one last amusing example, you might try to figure out the radius of convergence of the series $\sum_{n=1}^{\infty} f_n x^n$, where f_n are the Fibonacci numbers!