Lecture 37: Operations on power series

Last time we discovered that the domain of convergence of a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ is an interval — either $(-\infty, \infty)$, $(a-R, a+R)$ [+ possibly one or both endpoints], or $\{a\}$. The second big question is, what function $f(x)$ is the sum of the power series (on this domain)? In this lecture we will focus on the case $\sum c_n x^n$ (where $a=0$).

Begin with the geometric series

$$(1) \quad \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots = \frac{1}{1-x} \quad (\text{on } (-1,1)).$$

Now take the derivative (and reindex $n \mapsto n+1$):

$$(2) \quad \sum_{n=0}^{\infty} (n+1) x^n = 1 + 2x + 3x^2 + \dots = \frac{1}{(1-x)^2}.$$

Or, integrate from 0: applying $\int_0^x dx$ to $\sum_{n=0}^{\infty} t^n = \frac{1}{1-t}$ gives

$$\sum_{n=0}^{\infty} \int_0^x t^n dt = \int_0^x \frac{dt}{1-t} \qquad [\text{+ reindexing } n \mapsto n-1]$$

hence

$$(3) \quad \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots = -\ln(1-x)$$

(which actually holds on $[-1,1)$).

This makes sense, as for small $x>0$, $\ln(1-x)<0$.

If we substitute $x \mapsto -x$ and multiply by (-1), this gives

$$(4) \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots = \ln(1+x).$$

On the other hand, plugging $-x^2$ in for x in (1) yields

$$\sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \cdots = \frac{1}{1+x^2}$$

and integrating this (with t substituted for x) from 0 to x $\Rightarrow$

$$(5) \quad \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \arctan(x).$$

(i.e. $\tan^{-1}(x)$).

valid on $[-1, 1]$.

Ex 1/ Find a power series representation of $\ln(3+2x)$, and determine the radius of convergence.

Begin by differentiating: $\frac{d}{dx} \ln(3+2x) = \frac{2}{3+2x} = \frac{2}{3} \cdot \frac{1}{1+\frac{2}{3}x}$.

Since $\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$, plugging in $-\frac{2}{3}x = t$ gives

$$\frac{1}{1+\frac{2}{3}x} = \sum_{n=0}^{\infty} \left(-\frac{2}{3}\right)^n x^n, \quad \text{whereupon integrating } \int dx$$

$$\Rightarrow \ln(3+2x) = C + \sum_{n=0}^{\infty} \left(-\frac{2}{3}\right)^n \frac{x^{n+1}}{n+1} \underset{(n \mapsto n-1)}{=} C + \sum_{n=1}^{\infty} \left(-\frac{2}{3}\right)^{n-1} \frac{x^n}{n}.$$

Plugging in $x=0$, we find that $C = \ln(3)$. The radius of convergence can be found by the ratio test; alternatively, we could just note that $\sum t^n$ converges for $|t| < 1$, which is to say $\left|-\frac{2}{3}x\right| < 1$ or $|x| < \frac{3}{2}$. So $R = \frac{3}{2}$. //

Ex 2/ Find the sum of $\sum_{n=1}^{\infty} n(n+1)\left(\frac{1}{2}\right)^n$.

Consider the sum $S(x) = \sum_{n=1}^{\infty} n(n+1) x^n$. Dividing by x

and integrating gives $\int_0^y \frac{S(t)}{t}\,dt = \sum_{n=1}^{\infty} n(n+1)\int_0^y t^{n-1}\,dt$

$= \sum_{n=1}^{\infty}(n+1)y^n$, whereupon integrating <u>again</u> gives

$$\int_0^x\int_0^y \frac{S(t)}{t}\,dt\,dy = \sum_{n=1}^{\infty}(n+1)\int_0^x y^n\,dy = \sum_{n=1}^{\infty} x^{n+1} = \frac{x^2}{1-x}.$$

<u>Now</u> differentiate $\left(\frac{d}{dx}\right)$ to get $\int_0^x \frac{S(t)}{t}\,dt = \frac{(1-x)2x - x^2(-1)}{(1-x)^2} = \frac{2x-x^2}{(1-x)^2}$

and again to get $\frac{S(x)}{x} = \frac{2}{(1-x)^3} \Rightarrow S(x) = \frac{2x}{(1-x)^3}$.

So finally, plugging in $\frac{1}{2}$ yields $S\left(\frac{1}{2}\right) = \frac{2\cdot\frac{1}{2}}{(1-\frac{1}{2})^3} = \frac{1}{(\frac{1}{2})^3} = 8$. //

$\boxed{\text{TRY}}$ Find the sum of $\sum_{n=1}^{\infty} n\left(\frac{1}{3}\right)^n$.

<u>Ans</u>: Use $S(x) = \sum_{n=1}^{\infty} n x^n \Rightarrow \frac{S(x)}{x} = \sum_{n=1}^{\infty} n x^{n-1}$

$$\Rightarrow \int_0^x \frac{S(t)}{t}\,dt = \sum_{n=1}^{\infty} x^n = \frac{x}{1-x} \xRightarrow{d/dx} \frac{S(x)}{x} = \frac{(1-x)-x(-1)}{(1-x)^2}$$

$$= \frac{1}{(1-x)^2} \Rightarrow S\left(\frac{1}{3}\right) = \frac{1/3}{(1-1/3)^2} = \frac{1/3}{(2/3)^2} = \frac{1}{\cancel{3}}\cdot\frac{3^{\cancel{2}}}{2^2} = \frac{3}{4}.$$

The general statement underlying our calculations above is that if $S(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$ on an interval I, then for all x interior to I,

$$S'(x) = \sum_{n=1}^{\infty} n c_n x^n = c_1 + 2c_2 x + 3c_3 x^2 + \cdots$$

and $\int_0^x S(t)\,dt = \sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1} = c_0 + \frac{c_1}{2}x^2 + \frac{c_2}{3}x^3 + \cdots$.

That is, <u>we can differentiate & integrate termwise</u>.

Ex 3 / What is the sum $S(x)$ of $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$?

Let's try differentiating it:

$$S'(x) = 0 + 1 + \frac{2x}{2} + \frac{3x^2}{6} + \frac{4x^3}{24} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots = S(x).$$

Another function with this property is e^x: i.e., $\frac{d}{dx} e^x = e^x$.

Let's compare them by taking their quotient: in particular,

$$\frac{d}{dx} \frac{S(x)}{e^x} = \frac{e^x S'(x) - S(x) \frac{d}{dx} e^x}{e^{2x}} = \frac{e^x S(x) - S(x) e^x}{e^{2x}} = 0$$

$$\Rightarrow \frac{S(x)}{e^x} = C \text{ (constant)}, \text{ and } C = \frac{S(0)}{e^0} = \frac{1}{1} = 1.$$

$$\Rightarrow S(x) = e^x. \quad //$$

Ex 4 / Find the first few terms of a power series representation of $e^x \ln(1+x)$ and $\frac{\ln(1+x)}{e^x}$.

Power series behave like polynomials under addition, subtraction, multiplication, & division. For the product,

$$e^x \cdot \ln(1+x) = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right) \times \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)$$

$$= x + \left(1 - \frac{1}{2}\right) x^2 + \left(\frac{1}{2} - \frac{1}{2} + \frac{1}{3}\right) x^3 + \left(\frac{1}{6} - \frac{1}{4} + \frac{1}{3} - \frac{1}{4}\right) x^4 + \cdots$$

$$= \boxed{x + \frac{x^2}{2} + \frac{x^3}{3} + 0x^4 + \cdots}$$

(More generally, $\sum_{k=0}^{\infty} b_k x^k \cdot \sum_{l=0}^{\infty} c_l x^l = \sum_{n=0}^{\infty} \left(\sum_{k+l=n} b_k c_l \right) x^n$.)

For the quotient, we do long-division:

$$\boxed{x-\frac{3}{2}x^2+\frac{4}{3}x^3-x^4+\dots}$$

$$1+x+\frac{x^2}{2}+\frac{x^3}{6}+\dots \overline{\Big)\, x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots}$$

$$\begin{array}{r}
x+x^2+\frac{x^3}{2}+\frac{x^4}{6}+\dots \\
\hline
-\frac{3}{2}x^2-\frac{1}{6}x^3-\frac{5}{12}x^4+\dots \\
-\frac{3}{2}x^2-\frac{3}{2}x^3-\frac{3}{4}x^4+\dots \\
\hline
\frac{4}{3}x^3+\frac{1}{3}x^4+\dots \\
\frac{4}{3}x^3+\frac{4}{3}x^4+\dots \\
\hline
-x^4+\dots
\end{array}$$

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The general statement on intervals of convergence for these operations is that for $f(x)=\sum c_n x^n$, $g(x)=\sum b_n x^n$ both converging for $|x|<R$, the same is true for their sum, difference, and product. For the quotient $f(x)/g(x)$ (assuming $b_0\neq 0$) the radius of convergence may shrink. For example, any polynomial is its own power series on $(-\infty,\infty)$, but $\frac{1}{1-x}=1+x+x^2+\dots$ only has radius of convergence $=1$.

TRY Find the coefficient of x^3 in the power series representing $\frac{e^x}{2+x}$.

Ans: $\frac{1}{2+x}=\frac{1}{2}\cdot\frac{1}{1+\frac{x}{2}}=\frac{1}{2}\cdot\left(1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\dots\right)$

and so $\frac{e^x}{2+x}=\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\dots\right)\cdot\frac{1}{2}\cdot\left(1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\dots\right)$

$=\frac{1}{2}\left\{1+\left(1-\frac{1}{2}\right)x+\left(\frac{1}{2}-\frac{1}{2}+\frac{1}{4}\right)x^2+\left(\frac{1}{6}-\frac{1}{4}+\frac{1}{4}-\frac{1}{8}\right)x^3+\dots\right\}$

$$= \frac{1}{2}\left\{1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{24}+\cdots\right\} = \frac{1}{2}+\frac{x}{4}+\frac{x^2}{8}+\frac{x^3}{48}+\cdots .$$

So, $\frac{1}{48}$ is the coeff. of x^3.

Here's one final amusing example.

Ex 5/ Evaluate $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n(2n+1)} = 1-\frac{1}{9}+\frac{1}{45}-\frac{1}{189}+\cdots$.

Observe that $\arctan(x) = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$ for $|x|<1$. So

$$\frac{\pi}{6} = \arctan\left(\frac{1}{\sqrt{3}}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)\cdot\sqrt{3}^{2n+1}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)3^n\sqrt{3}}$$

$$\underset{\left(\substack{\text{mult.}\\ \text{by } \sqrt{3}}\right)}{\Longrightarrow} \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)3^n} = \frac{\pi\sqrt{3}}{6} .$$

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