Lecture 38: Taylor's formula

Last time we discovered that by differentiating and integrating power series term by term, we could construct power series representing many kinds of functions by starting with geometric series and being "clever" with these operations. We found:

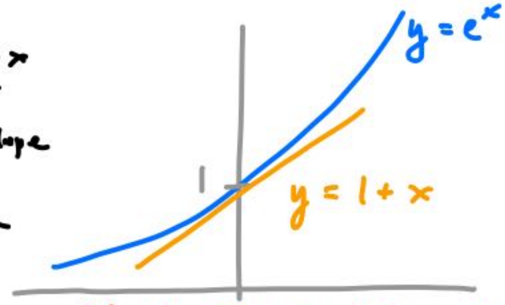
- $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$
- $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=0}^{\infty} (n+1)x^n$
- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$
- $\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$
- $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

[We can do even more if we include multiplication and (long) division of power series; see Example 4 from Lecture 37.] But this may not work to produce power series for every function, let alone a general formula.

Suppose we wanted to approximate $f(x) = e^x$ by a polynomial $T_n(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$ near 0.

<u>$n=1$</u>: the best choice is the tangent line:

$$\left.\begin{array}{l} f(0) = e^0 = 1 \\ f'(0) = e^0 = 1 \end{array}\right\} \Rightarrow T_1(x) = \underbrace{1}_{\text{y-intercept}} + \underbrace{1}_{\text{slope}} \cdot x$$



Since f is differentiable, we have

$$0 = \lim_{x \to 0} \frac{f(x) - f(0) - f'(0)x}{x}$$

$e^{0.1} \approx 1.105170918$, $1 + 0.1 = 1.1$

$$= \lim_{x \to 0} \frac{e^x - T_1(x)}{x}.$$ For any other choice of T_1, this wouldn't hold.

<u>$n=2$</u>: To get a better approximation, use a parabola instead of a line. We want $T_2(x)$ to have $T_2(0) = f(0)$, $T_2'(0) = f'(0)$, &

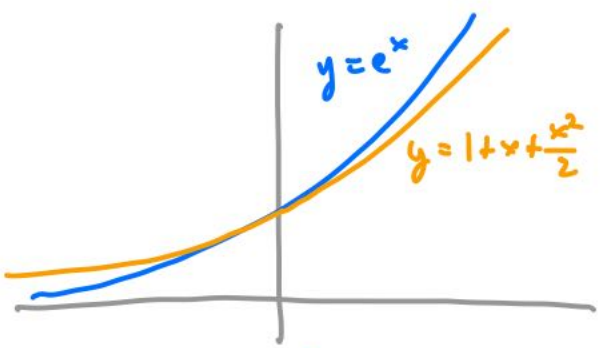
$\Rightarrow c_0 = 1$ $\qquad \Rightarrow c_1 = 1$

$T_2''(0) = f''(0) \Rightarrow 2c_2 = e^0 = 1 \Rightarrow c_2 = \frac{1}{2}$.

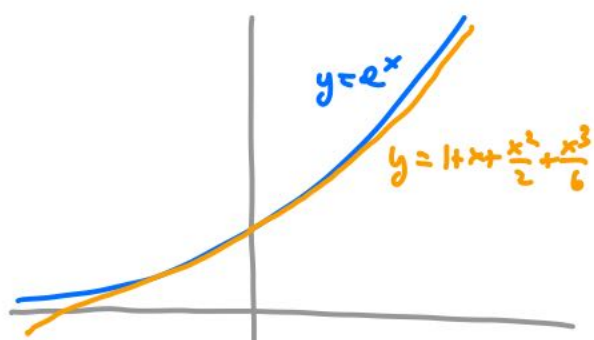
So $T_2(x) = 1 + x + \frac{x^2}{2}$. We have

$$\lim_{x \to 0} \frac{e^x - T_2(x)}{x^2} \overset{L}{=} \lim_{x \to 0} \frac{e^x - 1 - \cancel{\frac{2x}{2}}}{2x} = 0 \text{ by the above.}$$

<u>$n=3$</u>: If we want $T_3'''(0) = f'''(0)$ this gives $6c_3 = e^0 = 1$
$\Rightarrow c_3 = \frac{1}{6}$. So $T_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$.



$1 + 0.1 + \frac{(0.1)^2}{2} = 1.105$



$1 + 0.1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{6} = 1.1051666\ldots$

Hopefully you see a pattern developing! If we want $T_n(x)$ to approximate e^x, taking n derivatives of $c_n x^n$ gives $n! c_n$, whereas taking n derivatives of e^x gives e^x ($=1$ at $x=0$). So $c_n = \frac{1}{n!}$

seems like the best choice. This might lead you to suspect that e^x has power series representation $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$, which we already know by a different route.

Big Question: Given a function f, can we represent it as a power series in x, or (more generally) $x-a$?

Suppose such a representation exists:

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots$$

Then

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$$

$$f''(x) = \underbrace{1\cdot 2}_{2!}c_2 + 2\cdot 3c_3(x-a) + 3\cdot 4c_4(x-a)^2 + 4\cdot 5c_5(x-a)^3 + \cdots$$

$$f'''(x) = \underbrace{1\cdot 2\cdot 3}_{3!}c_3 + 2\cdot 3\cdot 4c_4(x-a) + 3\cdot 4\cdot 5c_5(x-a)^2 + \cdots$$

$$f^{(4)}(x) = \underbrace{1\cdot 2\cdot 3\cdot 4}_{4!}c_4 + 2\cdot 3\cdot 4\cdot 5c_5(x-a) + \cdots$$

and substituting $x=a$ gives

$$f'(a) = c_1 \quad , \quad f''(a) = 2!\,c_2 \Rightarrow c_2 = \frac{f''(a)}{2!} \, ,$$

$$f'''(a) = 3!\,c_3 \Rightarrow c_3 = \frac{f'''(a)}{3!} \, ,$$

or more generally $c_n = \frac{f^{(n)}(a)}{n!}$. (Note that $c_0 = \frac{f^{(0)}(a)}{0!} = f(a)$.)

This yields the

$\boxed{\text{Uniqueness Theorem}}$ **If** $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for all x in some interval around a, then $c_n = \frac{f^{(n)}(a)}{n!}$.

(Thus, f cannot have more than one power series in $(x-a)$ that represents it.)

Let f be a function with derivatives of all orders in some interval $(a-r, a+r)$. Then we can form the **Taylor series**

$$\sum_{n=0}^{\infty} \boxed{\frac{f^{(n)}(a)}{n!}}(x-a)^n \quad \longleftarrow n^{\text{th}} \text{ coefficient}$$

and ask whether this represents $f(x)$ (if any power series does, by the uniqueness theorem it's this one!). The partial sums of the series are called **Taylor polynomials**:

$$T_n(x) := \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k. \qquad (n^{\text{th}} \text{ Taylor poly.})$$

(If $a=0$ they are called **Maclaurin polynomials**, and the series is called the **Maclaurin series**.)

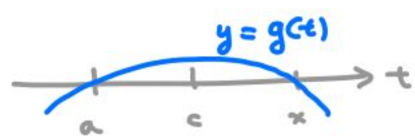
Define the **n^{th} remainder** by

$$R_n(x) := f(x) - T_n(x).$$

The Taylor series represents f in some interval $\iff$ $\lim_{n\to\infty} R_n(x) = 0$ in that interval. Thinking of x as constant, define

$$g(t) := f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k - R_n(x)\frac{(x-t)^{n+1}}{(x-a)^{n+1}}.$$

We have $g(x) = 0$, and also $g(a) = f(x) - T_n(x) - R_n(x) = 0$.



Applying the MVT to g, we get $c \in (a,x)$ such that $g'(c) = 0$.

But
$$g'(t) = 0 - \sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!}(x-t)^k + \underbrace{\sum_{k=1}^{n} \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1}}_{= \sum_{k=0}^{n-1} \frac{f^{(k+1)}(t)}{k!}(x-t)^k} + R_n(x)(n+1)\frac{(x-t)^n}{(x-a)^{n+1}}$$

$$= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + R_n(x)(n+1)\frac{(x-t)^n}{(x-a)^{n+1}}$$

$$\Rightarrow\quad 0 = g'(c) = -\frac{f^{(n+1)}(c)}{n!}(x-c)^n + R_n(x)(n+1)\frac{(x-c)^n}{(x-a)^{n+1}}$$

$$\Rightarrow\quad \boxed{R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}$$

Taylor's formula for the remainder

where c is some point between x & a. That is,

$$(*)\quad \boxed{|R_n(x)| \le \frac{\max_{t\in[a,x]}|f^{(n+1)}(t)|}{(n+1)!}|x-a|^{n+1}.}$$

Taylor's inequality

To conclude (for now), let's show that for $f(x) = e^x$, f = its Maclaurin series — i.e.

$$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}.$$

We must check that (for any x) $\lim_{n\to\infty} R_n(x) = 0$.

Now $a=0$, $f^{(n+1)}(t) = e^t$, and $\max_{t\in[0,x]} e^t = e^x$.

So $(*)$ $\Rightarrow$

$$|R_n(x)| \le \frac{e^x}{(n+1)!}|x|^{n+1} \xrightarrow[(n\to\infty)]{} 0,$$

and indeed $R_n(x) \to 0$.

[<u>Remark</u>: $\frac{r^n}{n!} \to 0$ for any real # r, since $\sum \frac{r^n}{n!}$ is a convergent series by the ratio test:

$$\rho = \lim_{n\to\infty}\frac{r^{n+1}/(n+1)!}{r^n/n!} = \lim_{n\to\infty}\frac{r}{n+1} = 0.\ ]$$