Lecture 39: Taylor & Maclaurin series

Let's begin by reviewing the outcome of our work in Lecture 38.

- Let f be a function with derivatives of all orders in some interval $I=(a-r, a+r)$. The Taylor series of f at a is

(1) $$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n .$$

- If any power series in $(x-a)$ converges to f on I, it has to be this one. (But this doesn't yet say this series converges, or that it converges to f.)
- The n^{th} Taylor polynomial of f at a is

(2) $$T_n(x) := \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k ,$$

and the n^{th} remainder is $R_n(x) := f(x) - T_n(x)$. For (1) to represent (converge/sum to) f on I is the same as having $\lim_{n\to\infty} R_n(x) = 0$ on I.

- Taylor's formula for the remainder says that

(3) $$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some point c between a & x. So if M = maximum value of $|f^{(n+1)}(x)|$ between x & a, then

(4) $$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} .$$

- If $a = 0$, then substitute "Maclaurin" for "Taylor" in (1) & (2).

Ex 1/ Find the Maclaurin series for $\sin(x)$ and show that it represents $\sin(x)$ for all x.

We first calculate the series coefficients in (1) $(a=0)$.

$$\begin{aligned}
f(x) &= \sin(x) &&\longrightarrow f(0) = 0\\
f'(x) &= \cos(x) &&\longrightarrow f'(0) = 1\\
f''(x) &= -\sin(x) &&\longrightarrow f''(0) = 0\\
f'''(x) &= -\cos(x) &&\longrightarrow f'''(0) = -1\\
f^{(4)}(x) &= \sin(x) &&\longrightarrow f^{(4)}(0) = 0\\
&\vdots && \qquad\quad \vdots
\end{aligned}$$

The Maclaurin series is therefore

$$(5) \qquad x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!},$$

and the remainder is bounded by

$$|R_n(x)| \leq \frac{\max_{t\in[0,x]} |f^{(n+1)}(t)|}{(n+1)!} |x|^{n+1} \leq \frac{|x|^{n+1}}{(n+1)!}$$

Since $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$ for all x, $\lim_{n\to\infty} R_n(x) = 0$ for all x,

and so (5) converges to $\sin(x)$ everywhere. //

Ex 2/ Same thing for $\cos(x) = f(x)$.

Differentiate the series for $\sin(x)$!

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}$$

$$\frac{d}{dx}\Big\downarrow \quad \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!}. \quad //$$

Remark: You may have heard of complex numbers, $a+bi$ where a & b are real numbers and $i = \sqrt{-1}$. Consider the power series $e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!} = \sum_{m=0}^{\infty} \frac{y^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{y^{2m+1}}{(2m+1)!}$ and now substitute $y = ix$:

$$e^{ix} = \sum_{m=0}^{\infty} \frac{(ix)^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{(ix)^{2m+1}}{(2m+1)!} = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} + i \cdot \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}$$

$= \cos(x) + i \cdot \sin(x)$. This formula was discovered by Euler and is of basic importance in physics & engineering.

Binomial Series

Recall that for any positive integer p,

$$(1+x)^p = 1 + \binom{p}{1} x + \binom{p}{2} x^2 + \cdots + \binom{p}{p} x^p,$$

where $\binom{p}{k} = \frac{p!}{k!(p-k)!} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}$

are the binomial coefficients appearing in Pascal's triangle.

Let $f(x) = (1+x)^p$, with p any real number.

Then $f(0) = 1$ while

$$f'(x) = p(1+x)^{p-1} \longrightarrow f'(0) = p$$

$$f''(x) = p(p-1)(1+x)^{p-2} \longrightarrow f''(0) = p(p-1)$$

$$f'''(x) = p(p-1)(p-2)(1+x)^{p-3} \longrightarrow f'''(0) = p(p-1)(p-2)$$

$$\vdots$$

$$f^{(n)}(x) = p(p-1)\cdots(p-n+1)^{p-n} \longrightarrow f^{(n)}(0) = p(p-1)\cdots(p-n+1).$$

Defining $\binom{p}{n} = \frac{p(p-1)\cdots(p-n+1)}{n!}$ for any real # p, the Maclaurin series for f is therefore

(6) $$\sum_{n=0}^{\infty} \binom{p}{n} x^n = 1 + px + \frac{p(p-1)}{2} x^2 + \frac{p(p-1)(p-2)}{6} x^3 + \cdots.$$

This doesn't converge on the whole real line: the ratio test gives

$$\rho = \lim_{n\to\infty}\left|\frac{\cancel{p(p-1)\cdots(p-n+1)}(p-n)\;\cancel{n!}}{\underset{n+1}{\cancel{(n+1)!}}\;\cancel{p(p-1)\cdots(p-n+1)}}\cdot\frac{x^{\cancel{n+1}}}{\cancel{x^{n}}}\right| = \lim_{n\to\infty}\frac{|p-n|\,|x|}{n+1} = |x|$$

$\Rightarrow$ radius of convergence $= 1$. I won't show that $R_n(x) \to 0$, but the series (6) does converge to $(1+x)^p$ on $(-1,1)$ (and at 1 if $p > -1$, and at -1 too if $p \geq 0$).

Ex 3/ Represent $\frac{1}{(1-x)^2}$ in a Maclaurin series for $x \in (-1,1)$.

$$(1-x)^{-2} = \sum_{n=0}^{\infty}\binom{-2}{n}x^n = 1 - 2x + \frac{(\cancel{-2})(-3)}{\cancel{2!}}x^2 + \frac{(\cancel{-2})(\cancel{-3})(-4)}{\cancel{3!}}x^3$$

$$= 1 - 2x + 3x^2 - 4x^3 + \ldots = \sum_{n=0}^{\infty}(-1)^n(n+1)x^n .$$

This should look familiar!

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Ex 4/ Represent $\sqrt{1+x}$ in a Maclaurin series and use it to approximate $\sqrt{1.1}$ to 5 decimal places.

$$(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!}x^4 + \cdots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \ldots$$

$$\sqrt{1.1} = \underbrace{1 + \frac{0.1}{2} - \frac{0.01}{8} + \frac{0.001}{16}}_{\text{since alternating series, this is already within of } \sqrt{1.1}} - \underbrace{\frac{5(0.0001)}{128}}_{< 0.000005} + \cdots$$

$$\approx 1.04881 .$$

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Ex 5/ Compute $\int_0^{0.4}\sqrt{1+x^4}\,dx$ to 5 decimal places.

We have $(1+x^4)^{1/2} = 1 + \frac{1}{2}x^4 - \frac{1}{8}x^8 + \frac{1}{16}x^{12} - \frac{5}{128}x^{16} + \cdots$

$$\Rightarrow \int_0^{0.4} \sqrt{1+x^4}\,dx = \left[x + \frac{x^5}{10} - \frac{x^9}{72} + \cdots\right]_0^{0.4}$$

$$\approx 0.4 + \frac{(0.4)^5}{10} = 0.\underbrace{401024}.$$

$\nwarrow$ $\frac{(0.4)^9}{72} < 0.000005$ //

Ex 6/ Find the Taylor series of e^x at $a=1$.

$f(x) = e^x \Rightarrow f^{(n)}(x) = e^x \Rightarrow f^{(n)}(1) = e^1 = e$ (for all n)

$$\Rightarrow e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=0}^{\infty} \frac{e}{n!}(x-1)^n.$$ //

TRY Find the coefficient of $(x-\frac{\pi}{6})^3$ in the Taylor series of $\sin(x)$ at $a=\frac{\pi}{6}$.

[Ans: $\frac{\sin'''(\pi/6)}{3!} = \frac{-\cos(\pi/6)}{6} = -\frac{\sqrt{3}}{12}$.]

Ex 7/ Find $\lim_{x\to 0} \frac{\sin(x) - x + \frac{x^3}{6}}{x^5}$.

You could use L'Hôpital repeatedly, or you could use power series: $\lim_{x\to 0} \frac{\left(\cancel{x} - \cancel{\frac{x^3}{3!}} + \frac{x^5}{5!} - \cdots\right) - \cancel{x} + \cancel{\frac{x^3}{6}}}{x^5} = \frac{1}{5!} = \frac{1}{120}$. //

TRY Use power series to find $\lim_{x\to 0} \frac{\sqrt{1+x} - 1 - \frac{1}{2}x}{x^2}$.

[Ans: $= \lim_{x\to 0} \frac{\left(\cancel{1} + \cancel{\frac{1}{2}x} - \frac{1}{8}x^2 + \cdots\right) - \cancel{1} - \cancel{\frac{1}{2}x}}{x^2} = -\frac{1}{8}$.]

(optional)
Remark: An easier way to show $(1+x)^p = \sum_{n=0}^{\infty} \binom{p}{n} x^n$ is to set $F(x) := \sum \binom{p}{n} x^n$, then notice that

$$(1+x)F'(x) = (1+x)\sum_{n=1}^{\infty} \binom{p}{n} n x^{n-1}$$

$$= (1+x)\sum_{n=1}^{\infty} p \binom{p-1}{n-1} x^{n-1}$$

$$\left[n\binom{p}{n} = n\,\frac{p(p-1)\cdots(p-n+1)}{n(n-1)!} = p\,\frac{(p-1)\cdots(p-n+1)}{(n-1)!} = p\binom{p-1}{n-1} \right]$$

$$= \sum_{n=1}^{\infty} p\binom{p-1}{n-1} x^{n-1} + \sum_{n=1}^{\infty} p\binom{p-1}{n-1} x^{n}$$

$$= \sum_{n=0}^{\infty} p\binom{p-1}{n} x^{n} + \sum_{n=1}^{\infty} p\binom{p-1}{n-1} x^{n}$$

$$= p\sum_{n=0}^{\infty} \left\{ \binom{p-1}{n} + \binom{p-1}{n-1} \right\} x^n \underset{\uparrow}{=} p\sum_{n=0}^{\infty} \binom{p}{n} x^n$$

use Pascal's triangle

$$= p\,F(x).$$

Now notice that if $f(x) = (1+x)^p$, then

$$(1+x)f'(x) = (1+x)\,p\,(1+x)^{p-1} = p(1+x)^p = p\,f(x).$$

So $\frac{d}{dx}\left(\frac{F(x)}{f(x)}\right) = \frac{f(x)F'(x) - F(x)f'(x)}{f(x)^2} = \frac{p\big(f(x)F(x) - F(x)f(x)\big)}{(1+x)\,f(x)^2} = 0$

$\Rightarrow F(x) = \text{const} \times f(x)$, and this "const." is clearly 1.

This shows $F(x) = f(x)$.