Lecture 4: Areas & Antiderivatives

Let's begin with the function

\[ f(t) = t \]

Here \( x \) is fixed (for the time being) and we are interested in the area under the curve from 0 to \( x \):

\[ \int_0^x f(t) \, dt = \int_0^x t \, dt = \text{Area}(\triangle) = \frac{1}{2} \cdot \text{base} \cdot \text{height} = \frac{1}{2} x^2. \]

Next, let's try

\[ f(t) = t^2 \]

... only, let's do it with Riemann sums.
\[ \int_a^b f(t) \, dt = \int_a^b t^2 \, dt = \lim_{n \to \infty} \left( \frac{a}{n} \right) \sum_{i=1}^{n} \left( \frac{i}{n} \right)^2 \Delta x \]

\[ = \int_a^b x^3 \, dx = \lim_{n \to \infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \int_a^b \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \]

\[ = \frac{1}{n^3} \left\{ \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} \right\} = \frac{x^3}{3} \]

One more: take \( f(t) = e^t \)

and write

\[ \int_a^b f(t) \, dt = \int_a^b e^t \, dt = \lim_{n \to \infty} \frac{e^x}{n} \sum_{i=1}^{n} e^{x_i} \]

\[ = \lim_{n \to \infty} \frac{x}{n} \sum_{i=1}^{n} \left( e^{x_i} \right)^i = \lim_{n \to \infty} \frac{x}{n} \cdot \frac{e^{x+n} - e^x}{e^x - 1} = \ldots \to \]

\[ = \frac{x^e}{e} \]

if \( S = a + a^2 + a^3 + \ldots + a^n \), then

\[ aS = a^2 + a^3 + \ldots + a^n + a^{n+1} \]

\[ \Rightarrow (a-1)S = a^{n+1} - a \Rightarrow S = a \frac{a^n - 1}{a - 1} \]

Quick "proof":

\[ (i+1)^3 - i^3 = 3i^2 + 3i + 1 \]

\[ \sum_{i=1}^{n} (i+1)^3 - i^3 = 3 \sum_{i=1}^{n} i^2 + 3 \sum_{i=1}^{n} i + n \]

\[ \sum_{i=1}^{n} i^3 + 3n^2 + 3n = 3 \sum_{i=1}^{n} i^2 + 3n^2 + \frac{3n}{2} \]

\[ \sum_{i=1}^{n} i^3 + \frac{3n^2}{2} + \frac{1}{2} \]

\[ \sum_{i=1}^{n} i^3 \frac{(n+1)(2n+1)}{6} = \sum_{i=1}^{n} i^2 \]
\[ (\text{continued}) \]
\[ \ldots = (e^x - 1) \lim_{n \to \infty} \frac{1}{n} (\text{why?}) \]
\[ = e^x - 1. \]

\[ \text{What do these three examples have in common?} \]
\[ \text{That's right,} \quad \begin{cases} x^{\frac{1}{2}} \\ x^{\frac{1}{3}} \\ e^x \\ e^x - 1 \end{cases} \]
\[ \text{is an antiderivative of} \quad \begin{cases} e^t \\ e^x \end{cases} \]
\[ \text{— carefully chosen flukes or general phenomenon?} \]

\[ \text{In other words, if we define} \]
\[ F(x) := \int_a^x f(t) \, dt = \text{“area up to } x\text{”} \]
\[ \text{is } F \text{ an antiderivative of } f \text{? i.e. does} \]
\[ F'(x) = f(x) ? \]

\[ \text{Aside:} \quad \int_a^x f(t) \, dt \text{ is a function of } x, \text{ not of } t. \]
\[ \text{We're looking at area under } y=f(t) \text{ from } t=a \text{ to } t=x; \text{ this area depends on } x. \text{ Here } t \text{ is just the integral version of a dummy index — The “integration variable”}. \]
How should we think of the rate of change of area (under the curve) up to $x$, with respect to $x$?

Heuristically, \[
\frac{\Delta F}{\Delta x} \approx \frac{f(x) \cdot \Delta x}{\Delta x} = f(x).
\]

More precisely, you want to use the definition of the derivative:

\[
F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{\int_{a}^{x+h} f(t) \, dt - \int_{a}^{x} f(t) \, dt}{h} = \lim_{h \to 0} \frac{1}{h} \int_{x}^{x+h} f(t) \, dt.
\]

To go further we use the picture.
As \( h \to 0 \), the maximum (\( M \)) and minimum (\( m \)) values of \( f \) on \([x, x+h]\) limit to \( f(x)\) by continuity. So by the Squeeze Theorem, we get

\[
F'(x) = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t) \, dt = f(x),
\]

\( \Rightarrow \) \( \int_a^x f(t) \, dt \) is always an antiderivative of \( f(x) \)!!

For different choices of \( a \), you’ll get different antiderivatives. Say \( b > a \). Then

\[
\int_a^x f(t) \, dt = \int_a^b f(t) \, dt + \int_b^x f(t) \, dt;
\]

as you’ll recall, two functions that differ by a constant have the same derivative (and indeed all antiderivatives of \( f \) differ by a constant).
So we write at the

**Fundamental Theorem of Calculus (V.1)**

\[
\frac{d}{dx} \int_{a}^{x} f(t) \, dt = f(x).
\]

This says that "taking the slope function" and "taking the area up to x function" are "inverse" to each other — do one, then the other, and you get back essentially the same thing. But it’s not surprising if you think the rate of accumulation of area under a function is proportional to the height of the function!

**Ex/ Find** \( \frac{dy}{dx} \)

(a) \( y = \int_{-2}^{x} \frac{1}{u+3} \, du \).

(b) \( y = \int_{0}^{3x} (1+t^4) \, dt \).

† If you do (a) then (b), the result may differ from the original by a constant.
In (b), we're taking the right-hand limit of integration to move \( 3 \) times as fast (\( 3x \) rather than \( x \)). So you'd expect the area to increase \( 3 \) times as quickly, and that's right. But we should be more systematic here and use the \underline{Chain Rule}.

What is \( \frac{d}{dx} \int_0^{g(x)} f(t) \, dt \)? If \( F(u) \) means \( \int_0^u f(t) \, dt \), this is \( \frac{d}{dx} F(g(x)) \). Some goofy function \( g(x) \) is controlling how the right-hand limit of integration is being moved. The \underline{Chain Rule} gives

\[
\frac{d}{dx} \int_0^{g(x)} f(t) \, dt = \frac{d}{dx} F(g(x)) = F'(g(x)) \cdot g'(x)
\]

\[= f(g(x)) \cdot g'(x).\]

\[F' = f\]

\(\uparrow\) The (right- and left-hand) limits of integration are the endpoints \( a \) and \( b \) of the interval over which you are calculating a definite integral \( \int_a^b f(x) \, dx \).
In particular, we get
\[ \frac{d}{dx} \int_0^{3x} f(t) \, dt = 3 \cdot f(3x) . \]

**Warning:** \[ \frac{d}{dx} \int_0^x f(g(t)) \, dt = f(g(x)) \]

There is no chain rule here: \( f(g(x)) \) is not being differentiated; only the "area up to \( x \)" is.

**Example**
\[ y = \int_2^x \sqrt{3+u} \, du , \quad x > 0 . \]

\[ \frac{dy}{dx} = \frac{d}{dx} (x^2 - 2) \cdot \sqrt{3 + (x^2 - 2)} \]
\[ = 2x \cdot \sqrt{x^2 + 1} . \]

To formalize this, which may make it more or less comprehensible, write
\[ \int_2^{x^2 - 2} \sqrt{3+u} \, du = F(g(x)) \]
where
\[
\begin{align*}
F(z) &= \int_2^z \sqrt{3+u} \, du \quad (\text{has derivative } \frac{1}{\sqrt{3+z}}) \\
g(x) &= x^2 - 2 .
\end{align*}
\]
Then
\[ \frac{d}{dx} F(g(x)) = F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x) \]
\[ = 2x \cdot \sqrt{3 + g(x)} = 2x \sqrt{x^2 + 1} . \]
Ex: \( y = \int_{x}^{2x} (1+t^2) \, dt \). Find \( \frac{dy}{dx} \)!

We'll look on Friday at how the relationship between differentiation and integration discovered today makes computing integrals a whole lot more straightforward.