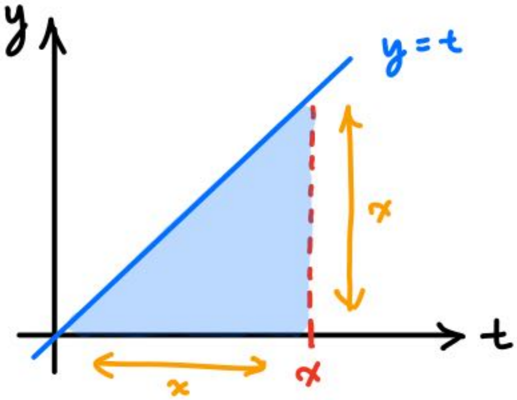
Lecture 4: Areas & Antiderivatives

Let's begin with the function

$$f(t) = t$$



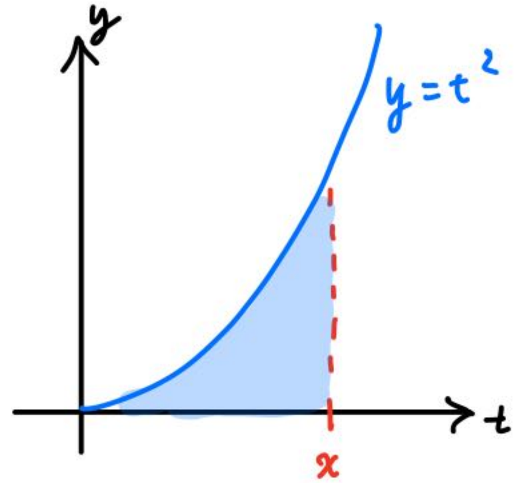
Here x is fixed (for the time being) and we are interested in the area under the curve from 0 to x:

$$\int_0^x f(t)\,dt = \int_0^x t\,dt = \text{Area}(\blacktriangle) = \frac{1}{2}\cdot\text{base}\cdot\text{height}$$
$$= \frac{1}{2}x^2.$$

Next, let's try

$$f(t) = t^2$$

... only, let's do it with Riemann sums.

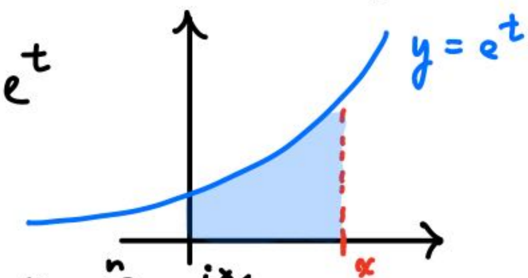


$$\int_0^x f(t)\,dt = \int_0^x t^2\,dt = \lim_{n\to\infty} \underbrace{\frac{x}{n}}_{\Delta x} \sum_{i=1}^{n} \Big(\underbrace{\frac{ix}{n}}_{x_i = i\Delta x}\Big)^2$$

$$= x^3 \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2 \underset{\dagger}{=} x^3 \lim_{n\to\infty} \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$= x^3 \lim_{n\to\infty} \left\{ \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} \right\} = \frac{x^3}{3}.$$

One more: take $f(t) = e^t$



and write

$$\int_0^x f(t)\,dt = \int_0^x e^t\,dt = \lim_{n\to\infty} \frac{x}{n} \sum_{i=1}^{n} e^{ix/n}$$

$$= \lim_{n\to\infty} \frac{x}{n} \sum_{i=1}^{n} \left(e^{x/n}\right)^i = \lim_{n\to\infty} \frac{x}{n} e^{x/n} \frac{e^x - 1}{e^{x/n} - 1} = \cdots \longrightarrow$$

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if $S = a + a^2 + a^3 + \cdots + a^n$, then

$$aS = a^2 + a^3 + \cdots + a^n + a^{n+1}$$

$$\Rightarrow (a-1)S = a^{n+1} - a \Rightarrow S = a\,\frac{a^n - 1}{a - 1}$$

$\dagger$ quick "proof": apply $\sum_{i=1}^{n}$ to both sides

$$(i+1)^3 - i^3 = 3i^2 + 3i + 1$$

$$(n+1)^3 - 1^3 = 3\sum_{i=1}^{n} i^2 + 3\overbrace{\sum_{i=1}^{n} i}^{\frac{n(n+1)}{2}} + n$$

$$n^3 + 3n^2 + 3n = 3\sum_{i=1}^{n} i^2 + \frac{3}{2}n^2 + \frac{5}{2}n$$

$$n^3 + \frac{3}{2}n^2 + \frac{1}{2}n = 3\sum_{i=1}^{n} i^2$$

$$\frac{n(n+1)(2n+1)}{6} = \sum_{i=1}^{n} i^2$$

(continued)

$$\ldots = (e^x - 1) \cdot \cancel{\lim_{n\to\infty} e^{x/n}} \xrightarrow{} 1 \text{ (why?)} \cdot \lim_{n\to\infty} \frac{x/n}{e^{x/n}-1}$$

Same as $\lim_{z\to 0} \frac{z}{e^z - 1} \underset{\text{L'Hôpital}}{=} \lim_{z\to 0} \frac{1}{e^z} = 1$

$$= e^x - 1 .$$

What do these three examples have in common?

That's right, $\left.\begin{matrix} x^2/2 \\ x^3/3 \\ e^x - 1 \end{matrix}\right\}$ is an antiderivative of $\left\{\begin{matrix} x \\ x^2 \\ e^x \end{matrix}\right.$

— carefully chosen flukes or general phenomenon?

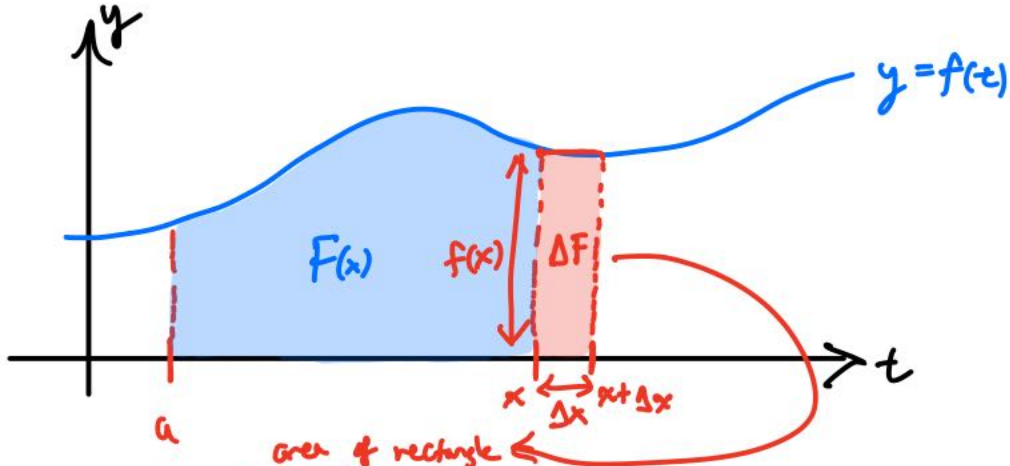
In other words, if we define

$$F(x) := \int_a^x f(t)\,dt = \text{"area up to } x\text{"}$$

is F an antiderivative of f? i.e. does $F'(x) = f(x)$?

Aside: $\int_a^x f(t)dt$ is a function of x, not of t. We're looking at area under $y = f(t)$ from $t = a$ to $t = x$; this area depends on x. Here t is just the integral version of a dummy index — the "integration variable".

How should we think of the rate of change of area (under the curve) up to x, with respect to x?



Heuristically $\dfrac{\Delta F}{\Delta x} \approx \dfrac{f(x)\cdot\Delta x}{\Delta x} = f(x)$.

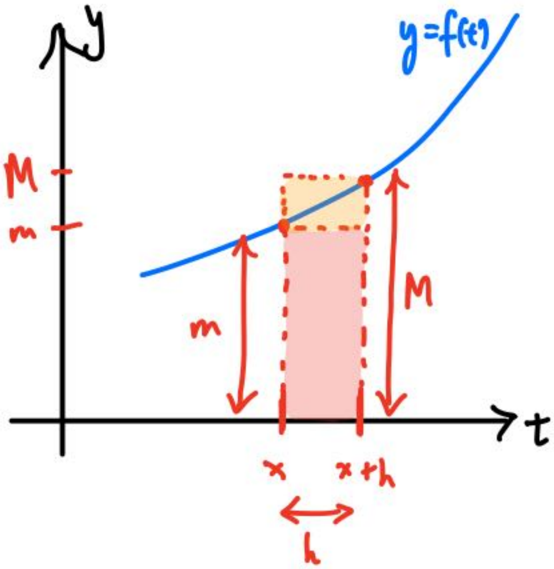
More precisely, you want to use the definition of the derivative:

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

To go further we use the picture



which gives

$$mh \le \int_x^{x+h} f(t)\,dt \le Mh$$

$$\Rightarrow \quad m \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le M$$

squeeze thm.

$f(x)$

As $h \to 0$, the maximum (M) and minimum (m) values of f on $[x, x+h]$ limit to $f(x)$ by continuity.

So by the Squeeze Theorem, we get

$$F'(x) = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x),$$

$\Longrightarrow$ $\boxed{\int_a^x f(t) \text{ is always an antiderivative of } f(x)\,!!}$

For different choices of $\underline{\underline{a}}$, you'll get different antiderivatives. Say $b > a$. Then

$$\int_a^x f(t)\,dt = \underbrace{\int_a^b f(t)\,dt}_{\text{constant}} + \int_b^x f(t)\,dt\,;$$

as you'll recall, two functions that differ by a constant have the same derivative (think: slope function) (& indeed all antiderivatives of f differ by a constant).

So we arrive at the

$$\boxed{\begin{array}{c}\text{FUNDAMENTAL THEOREM of CALCULUS (v. 1)} \\ \dfrac{d}{dx}\underbrace{\int_a^x f(t)\,dt}_{F(x)} = f(x).\end{array}}$$

This says that "$\underbrace{\text{taking the slope function}}_{(a)}$" and "$\underbrace{\text{taking the area-up-to-}x\text{ function}}_{(b)}$" are "inverse" to each other — do one, then the other, and you get back essentially† the same thing. But it's not surprising if you think: <u>the rate of accumulation of area under a function is proportional to the height of the function</u>!

Ex/ Find $\dfrac{dy}{dx}$ if (a) $y = \displaystyle\int_{-2}^{x} \frac{1}{u+3}\,du$.

(b) $y = \displaystyle\int_0^{3x} (1+t^4)\,dt$. //

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† If you do (a) then (b), the result may differ from the original by a constant.

In (b), we're taking the right-hand limit of integration† to move 3 times as fast ($3x$ rather than x). So you'd expect the area to increase 3 times as quickly, and that's right. But we should be more systematic here and use the Chain Rule.

What is $\frac{d}{dx}\int_0^{g(x)} f(t)\,dt$? If $F(u)$ means $\int_0^u f(t)\,dt$, this is $\frac{d}{dx} F(g(x))$. Some goofy function $g(x)$ is controlling how the right-hand limit of integration is being moved. The Chain Rule gives

$$\frac{d}{dx}\int_0^{g(x)} f(t)\,dt = \frac{d}{dx} F(g(x)) = F'(g(x)) \cdot g'(x)$$

$$= f(g(x)) \cdot g'(x).$$

$F' = f$

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† The (right- and left-hand) limits of integration are the endpoints a & b of the interval over which you are calculating a definite integral $\int_a^b f(x)\,dx$.

In particular, we get

$$\frac{d}{dx}\int_0^{3x} f(t)\,dt = 3\cdot f(3x).$$

$\boxed{\text{WARNING: } \quad \frac{d}{dx}\int_0^{x} f(g(t))\,dt = f(g(x)) \quad !!}$

There is no chain rule here: $f(g(t))$ is <u>not</u> being differentiated, only the "area up to x" is.

Ex/ $y = \int_2^{x^2-2} \sqrt{3+u}\,du$, $x > 0$.

$$\frac{dy}{dx} = \frac{d}{dx}(x^2-2)\cdot\sqrt{3+(x^2-2)}$$

$$= 2x\sqrt{x^2+1}.$$

To formalize this, which may make it more or less comprehensible, write

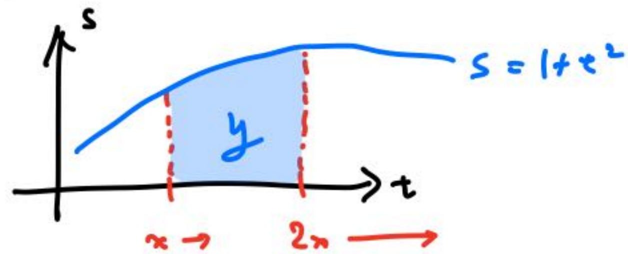
$$\int_2^{x^2-2}\sqrt{3+u}\,du = F(g(x)) \quad \text{where}$$

$$\begin{cases} F(z) = \int_2^{z}\sqrt{3+u}\,du & \left(\text{has derivative } \sqrt{3+z}\right) \\ \& \\ g(x) = x^2-2. \end{cases}$$

Then $\frac{d}{dx}F(g(x)) = F'(g(x))\cdot g'(x) = f(g(x))\cdot g'(x)$

$$= 2x\cdot\sqrt{3+g(x)} = 2x\sqrt{x^2+1}. \quad //$$

Ex/ $y = \int_x^{2x} (1+t^2)\, dt$. Find $\frac{dy}{dx}$!



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We'll look on Friday at how the relationship between differentiation and integration discovered today makes computing integrals a whole lot more straightforward.