Lecture 40: Applications of Taylor series

We have already seen a couple of applications, to estimating a square root and an integral to 5 decimal places. We'll begin with a few more examples of this type. First the

REVIEW

<u>Taylor series of f at a</u>: $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

<u>Taylor's Formula with Remainder</u>:

$$f(x) = \underbrace{f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}_{T_n(x) = \text{Taylor polynomial of } f \text{ at } a} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}_{\text{Remainder} = R_n(x)}$$

some number between a and x

<u>Some famous Maclaurin series</u>:

* $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

* $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ $\frac{d}{dx}$

$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

* $(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots + \binom{p}{n}x^n + \cdots$ p = any real number

* $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$ plug in $-x^2$, integrate

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ plug in $-x$, integrate

Ex 1/ Estimate $\sin(1)$ with a 3^{rd}-degree Taylor polynomial, and determine a bound on the accuracy of this estimate.

We use $a=0$ (Maclaurin polynomial $T_3(x) = x - \frac{x^3}{3!}$ for $\sin(x)$):

$$\sin(1) \approx 1 - \frac{1^3}{6} = \frac{5}{6} = 0.8333\ldots$$

The remainder/error is $|R_3(1)| = \frac{|\sin''''(c)|}{4!}|1|^4 \le \frac{1}{4!} = \frac{1}{24} < 0.042$.

That is, $0.833\ldots - 0.042 \le \sin(1) \le 0.833\ldots + 0.042$.

Since the series $1 - \frac{1^3}{3!} + \frac{1^5}{5!} - \ldots$ for $\sin(1)$ is alternating, the error may alternatively be bounded by the next term in the series: $\frac{1}{5!} = \frac{1}{120} = 0.00833\ldots$. This gives the better result

$$0.83333\ldots - 0.00833 \le \sin(1) \le 0.833333\ldots + 0.00833.$$ //

Ex 2/ Estimate $\sqrt{e}$ with a second-degree Maclaurin polynomial, and determine a bound for its accuracy; repeat w/degree 10 polynomial.

The 2^{nd} Maclaurin polynomial for e^x is $T_2(x) = 1 + x + x^2/2$.

$$\sqrt{e} = e^{0.5} \approx 1 + 0.5 + \frac{(0.5)^2}{2} = 1 + 0.5 + 0.125 = 1.625.$$

$$|R_2(0.5)| = \frac{e^c}{3!}(0.5)^3 \underset{c\in[0,0.5]}{\le} \frac{2}{3!}(0.5)^3 = \frac{1}{24} < 0.042.$$

The 10^{th} degree Maclaurin polynomial is $T_{10}(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{10}}{10!}$, and $T_{10}(0.5) = 1.648721271\ldots$, while $|R_{10}(0.5)| = \frac{e^c}{11!}(0.5)^{11} \le \frac{2}{11!}(0.5)^{11} < 0.00000000003 \Rightarrow T_{10}(0.5)$ is accurate to within 10 digits. (Note that we could not use the alternating series error bound here, because the series isn't alternating.) //

The Maclaurin series work fine for estimating a function $f(x)$ when x is near 0; but what if we wanted to estimate $\sqrt{8.9}$? It would be better to use a Taylor series for $x^{1/2}$

about $a = 9$, right? So let's do a couple simpler examples first:

Ex 3/ Find the Taylor series of e^x at $a = 1$.

$$f(x) = e^x \Rightarrow f^{(n)}(x) = e^x \Rightarrow f^{(n)}(1) = e^1 = e \quad (\text{for all } n)$$

$$\Rightarrow e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=0}^{\infty} \frac{e}{n!}(x-1)^n. \quad //$$

$\boxed{\text{TRY}}$ Find the coefficient of $(x - \frac{\pi}{6})^4$ in the Taylor series of $\sin(x)$ at $a = \frac{\pi}{6}$.

[Ans: $\dfrac{\sin''''(\pi/6)}{4!} = \dfrac{\sin(\pi/6)}{24} = \dfrac{1/2}{24} = \dfrac{1}{48}$.]

Ex 4/ Let $T_2(x)$ be the 2^{nd} Taylor polynomial of $f(x) = \sqrt{x}$ at $a = 9$. Estimate the accuracy of the approximation $f(x) \approx T_2(x)$ for x in the range $[8.9, 9.1]$.

We have $f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = -\frac{1}{4}x^{-3/2}$, $f'''(x) = \frac{3}{8}x^{-5/2}$

$\Rightarrow f(9) = 3$, $f'(9) = \frac{1}{6}$, $f''(9) = \dfrac{-1}{4 \cdot 3^3} = \dfrac{-1}{108}$

$\Rightarrow T_2(x) = 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2$, while for $x \in [8.9, 9.1]$

$$|R_2(x)| = \frac{|f'''(c)|}{3!}|x-9|^3 \leq \frac{\max_{t \in [8.9, 9.1]} \frac{3}{8}t^{-5/2}}{6} \cdot (0.1)^3$$

$= \dfrac{3}{8 \cdot (8.9)^{5/2}} \leq 0.002$

$\leq 0.0000039\ldots$

In particular, we could estimate $\sqrt{8.9}$ to 5 decimal places by $3 + \frac{1}{6}(-0.1) - \frac{1}{216}(-0.1)^2 \approx 2.983287\ldots$ //

Another application of Taylor series is to computing limits of $\frac{0}{0}$ type that would be horrible to do via L'Hôpital.

Ex 7/ Find $\displaystyle\lim_{x\to 0} \frac{\sin(x) - x + \frac{x^3}{6}}{x^5}$.

You could use L'Hôpital repeatedly, or you could use power series: $\displaystyle\lim_{x\to 0} \frac{\left(\cancel{x} - \cancel{\frac{x^3}{3!}} + \frac{x^5}{5!} - \dots\right) - \cancel{x} + \cancel{\frac{x^3}{6}}}{x^5} = \frac{1}{5!} = \frac{1}{120}$. //

$\boxed{\text{TRY}}$ Use power series to find $\displaystyle\lim_{x\to 0} \frac{(1+x)^{1/3} - 1 - \frac{1}{3}x}{x^2}$.

[Ans: $\displaystyle = \lim_{x\to 0} \frac{\left(\cancel{1} + \cancel{\frac{1}{3}x} + \frac{\frac{1}{3}\left(\frac{-2}{3}\right)}{2!}x^2 + \dots\right) - \cancel{1} - \cancel{\frac{1}{3}x}}{x^2} = \frac{-1}{9}$.]

Let's end this course on a high note. You may be familiar with Euclid's proof that $\sqrt{2}$ is irrational: if it were — that is, if it could be written as $\frac{P}{Q}$ with P & Q integers with no common prime factor. So then

$$\sqrt{2} = \frac{P}{Q} \implies Q\sqrt{2} = P$$
$$\implies Q^2 \cdot 2 = P^2$$
$$\implies P \text{ is even}$$
$$\implies 4 \text{ divides } P^2$$
$$\implies Q \text{ is even}$$
$$\implies P \,\&\, Q \text{ share 2 as factor: Contradiction!}$$

rational numbers are exactly the terminating or repeating decimals. So $\sqrt{2} = 1.414213\ldots$ doesn't repeat!

No calculus in that one.

What about e? Consider the two sequences of integers

$$a_n = n!\ , \quad b_n = \sum_{k=0}^{n} \frac{n!}{k!}\ ;$$

then $\displaystyle\frac{b_n}{a_n} = \sum_{k=0}^{n} \frac{1}{k!} = T_n(1)$, where $\displaystyle T_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$

is the n^{th} Maclaurin polynomial approximating e^x. So

$$0 < \left| e - \frac{b_n}{a_n} \right| = \left| e^1 - T_n(1) \right| = |R_n(1)| = \frac{e^c}{(n+1)!} 1^{n+1}$$

$$(\text{for } c \in [0,1]) \qquad \leq \frac{3}{(n+1)!} .$$

Now suppose $e = \frac{P}{Q}$ with P & Q integers. Then

$$\left| \frac{P}{Q} - \frac{b_n}{a_n} \right| \leq \frac{3}{(n+1)!} = \frac{3/(n+1)}{a_n}$$

multiply by Q and a_n

$$|a_n P - b_n Q| \leq \frac{3Q}{n+1}$$

For some large N, $\frac{3Q}{N+1} < 1$, and so

$$0 < |\underbrace{a_N P - b_N Q}_{\text{integer ... between } 0 \,\&\, 1\,??}| < 1$$

Contradiction!

Therefore e is irrational!

On Friday we will review for the (all multiple choice) Final Exam.