Lecture 41 : FINAL EXAM REVIEW

FORMAT: 20 Multiple-choice questions worth 35% of grade

FORMULAS GIVEN: $\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$, $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$, $\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$, and (if $T_n(x)$ is the n^{th} Taylor polynomial of f at $x=a$) $f(x) - T_n(x) =: R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$.

MATERIAL COVERED: 10 problems on sequences & series (7 of those on power series), 10 on prior material.

WHAT OF THE PRIOR MATERIAL IS ON THE EXAM?

Integrals:

- as the limit of a Riemann sum:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3e^{3i/n}}{n} = \int_0^3 e^x \, dx$$

- as the total change in the integrand's antiderivative:

e.g. power $P(t)$ is the rate at which energy is consumed by some appliance.

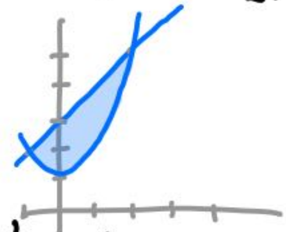
If $P(t) = 2 + \sin\left(\frac{\pi t}{12}\right)$ Watts (Joules/second), then the <u>total energy used</u> from time $t = 0$ sec. to $t = 6$ sec. is $\int_0^6 P(t)\,dt = \left[2t - \frac{12}{\pi}\cos\left(\frac{\pi t}{12}\right)\right]_0^6 = 12 + \frac{24}{\pi}$ J.

- as an antiderivative (via FTC):

$$\frac{d}{dx}\int_{2x}^{3x^2}\sqrt{15-t}\,dt = 6x\sqrt{15-3x^2} - 2\sqrt{15-2x}.$$

- as an area (under a curve, or between 2 [or more]):

e.g., between $y = 1+x^2$ and $y = 3+x$



to find intersection: $1+x^2 = 3+x$

$x^2 - x - 2 = 0$

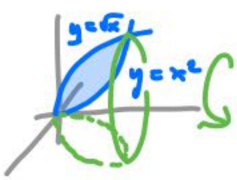
$(x-2)(x+1) = 0$

$$A = \int_{-1}^{2}\left[(3+x) - (1+x^2)\right]dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2} = \frac{9}{2}.$$

- as a probability or mean (expected/average value):

Given a density function $f(x)$ for a random variable X (i.e. $\int_{-\infty}^{\infty} f(x)\,dx = 1$), $P(a \le X \le b) = \int_a^b f(x)\,dx$ and $\mu = \int_{-\infty}^{\infty} x f(x)\,dx$.

- as a volume of a solid of revolution:



$$\left.\begin{aligned} \text{Washers: } & \int_0^1 \pi\left((\sqrt{x})^2 - (x^2)^2\right)dx \\ \text{Shells: } & \int_0^1 2\pi(y)\left(\sqrt{y} - y^2\right)dy \end{aligned}\right\} = \frac{3}{10}\pi$$

$x = \sqrt{y}$, $x = y^2$

<u>Techniques of integration</u>:

- substitution: $\int_1^e \frac{(\ln(x))^2}{x}\,dx \overset{\substack{u=\ln(x)\\ du = dx/x}}{=} \int_0^1 u^2\,du = \left.\frac{u^3}{3}\right|_0^1 = \frac{1}{3}$

- integration by parts: $\int_{\pi/2}^{\pi}\left(-x\cos(x)\right)dx$

$$\underset{\substack{\uparrow}}{=} -x\sin(x)\Big|_{\pi/2}^{\pi} + \int_{\pi/2}^{\pi} \sin(x)\,dx = \frac{\pi}{2} - \cos(x)\Big|_{\pi/2}^{\pi} = \frac{\pi}{2} + 1$$

$u = -x$, $dv = \cos(x)\,dx$

$du = -dx$, $v = \sin(x)$

- partial fractions: $\displaystyle\int_1^2 \frac{4x-11}{2x^2+7x-4}\,dx = \int_1^2 \frac{-2\,dx}{2x-1} + \int_1^2 \frac{3\,dx}{x+4}$

$$\left(\begin{array}{l} \dfrac{4x-11}{(2x-1)(x+4)} = \dfrac{A}{2x-1} + \dfrac{B}{x+4} = \dfrac{(A+2B)x + (4A-B)}{(2x-1)(x+4)} \\ \Rightarrow \begin{cases} 4 = A+2B \\ -11 = 4A-B \end{cases} \Rightarrow \begin{cases} 4 = A+2B \\ -22 = 8A-2B \end{cases} \Rightarrow -18 = 9A \Rightarrow A = -2 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Rightarrow B = 3 \end{array} \right)$$

$$= -\ln|2x-1|\Big|_1^2 + 3\ln|x+4|\Big|_1^2 = -\ln 3 + \underbrace{\cancel{\ln 1}}_{0} + 3\ln 6 - 3\ln 5$$

$$= \ln\left(\frac{6^3}{5^3 3}\right).$$

- trigonometric substitution: $\displaystyle\int_1^{\sqrt{3}} \frac{dx}{x^2\sqrt{4-x^2}}$
  (know $\sin^2 + \cos^2 = 1$,
  no $\frac{1}{2}$-angle formulas needed)

$$\Big\| \Leftarrow \quad x = 2\sin\theta, \quad dx = 2\cos\theta\,d\theta$$

$$\frac{1}{2\sqrt{3}} = \frac{-1}{4}\cot\theta\Big|_{\pi/6}^{\pi/3} = \int_{\pi/6}^{\pi/3} \frac{d\theta}{4\sin^2\theta} = \int_{\pi/6}^{\pi/3} \frac{2\cos\theta\,d\theta}{4\sin^2\theta\,\underbrace{\sqrt{4-4\sin^2\theta}}_{2\cos\theta}}$$

- improper integrals (see the review for Exam III)

- no trig integrals ($\int \sin^a \cos^b$ etc.)

WHAT ABOUT THE MATERIAL ON SEQUENCES + SERIES?

Sequences:

- recursively defined sequences: e.g. $a_1 = 5$, $a_{n+1} = \sqrt{4+a_n}$

  If $L = \lim_{n\to\infty} a_n$ exists, then $L = \sqrt{4+L}$ $\Rightarrow$

$L^2 = 4 + L \Rightarrow L^2 - L - 4 = 0 \Rightarrow L = \frac{1 \pm \sqrt{17}}{2}$.

Clearly must be $\frac{1+\sqrt{17}}{2}$ (a little over $5/2$). Also $a_n \geq 0$ (bounded below) and decreasing: $a_2 = 3$, $a_3 = \sqrt{7}$, ... (can prove by induction).

- compute limits of a_{n+1}/a_n for ratio test

Series:

- geometric series:
$$\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^{n+1}} = \frac{2^{1-1}}{3^{1+1}} + \frac{2^{2-1}}{3^{2+1}} + \cdots$$
$$= \frac{1}{9} + \frac{2}{3} \cdot \frac{1}{9} + \cdots$$
$$= \frac{1/9}{1 - 2/3} = \frac{1/9}{1/3} = \frac{1}{3}.$$

- ratio test: (a) for convergence of series of numbers
$$\sum_{n=0}^{\infty} \frac{(3n)!}{(n!)^3} \rightsquigarrow \rho = \lim_{n\to\infty} \frac{(3n+3)!}{(n+1)!^3} \Big/ \frac{(3n)!}{n!^3} = \frac{(3n+3)(3n+2)(3n+1)}{(n+1)^3}$$
$$= 27. \to \text{diverges}$$

  (b) for radius of convergence of power series
$$\sum_{n=0}^{\infty} \frac{(3n)!}{(n!)^3} (x-1)^n \rightsquigarrow \rho = \lim_{n\to\infty} \frac{(3n+3)(3n+2)(3n+1)}{(n+1)^3} |x-1|$$
$$= 27|x-1|$$
$\Rightarrow$ radius of conv. is $1/27$, and have AC in $(1 - \frac{1}{27}, 1 + \frac{1}{27})$.

- basic Maclaurin series: for $\sin(x)$, $\cos(x)$, $\frac{1}{1-x}$, $(1+x)^p$, e^x
  (a) construct new power series from these: e.g. first four terms of $\sqrt{1+x}\cos(x)$, $\ln(1-x)$, $\tan(x)$.
  (b) recognize evaluations of these power series at a point: $\sum (-1)^m \frac{\pi^{2m}}{16^m (2m)!} = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

(c) Use them to compute $\frac{0}{0}$ limits (cf. lecture 40)

- Taylor series of f at $x=a$ is $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$.

  - - polynomial - - - - - - $\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \left(= T_n(x)\right)$.

  Know these formulas!

- Be able to use Taylor's remainder formula to bound the error in using $T_n(x)$ to estimate $f(x)$.

If it hasn't been mentioned here, it isn't on the exam. Good luck!