Lecture 5: How to integrate

Let's review: let $f(x)$ be a function defined on $[a,b]$.

$f'(x^*)$ measures the rate-of-change of f at $x^* \in [a,b]$.

So $\Delta f \approx f'(x^*) \cdot \Delta x \quad \left(= \cancel{\frac{\Delta f}{\Delta x} \Delta x} \right) \ldots$ BUT:

$f'(x^*) \Delta x$ is <u>also</u> the area of a little box under f'.

Adding these up as x^* ranges from a to b gives

$$\begin{array}{ccccc} \sum_{i=1}^{n} f'(x_i^*) \Delta x & \approx & \sum_{i=1}^{n} (\Delta f)_i & = & \text{total change in } f \\ \big\downarrow {\scriptstyle n \to \infty} & & & & \big\| \\ \int_a^b f'(x)\, dx & & \overset{!}{=\!=\!=\!=\!=} & & f(b) - f(a) \end{array}$$

$\Rightarrow$ area under <u>derivative</u> f' = change in f.

$\Rightarrow$ area under f = change in antiderivative !

Let's approach this a bit differently:

given f, let F be <u>any</u> antiderivative. e.g.

- $f(x) = x^n \rightsquigarrow F(x) = \frac{x^{n+1}}{n+1}$
- $f(x) = \sin(x) \rightsquigarrow F(x) = -\cos(x)$.

Now recall

$\boxed{\text{FTC v.1.0}}$ $\quad \dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$; that is,

$\mathcal{F}(x) := \displaystyle\int_a^x f(t)\,dt$ is an antiderivative of $f(x)$.

$$\text{So } \mathcal{F}' = f = F' \implies (\mathcal{F} - F)' = f - f = 0$$

$$\implies \mathcal{F} - F = C \quad (\text{constant})$$

$$\implies \int_a^x f(t)\,dt = F(x) + C$$

$$\implies 0 = \int_a^a f(t)\,dt = F(a) + C \quad (\text{so } C = -F(a))$$

$$\implies \int_a^b f(t)\,dt = F(b) - F(a) =: \text{“} F(x)\Big|_a^b \text{”},$$

confirming that we can compute definite integrals just by finding the change in the antiderivative:

$$\text{e.g.} \quad \int_a^b x^n\,dx = \frac{x^{n+1}}{n+1}\Big|_a^b = \frac{b^{n+1} - a^{n+1}}{n+1}.$$

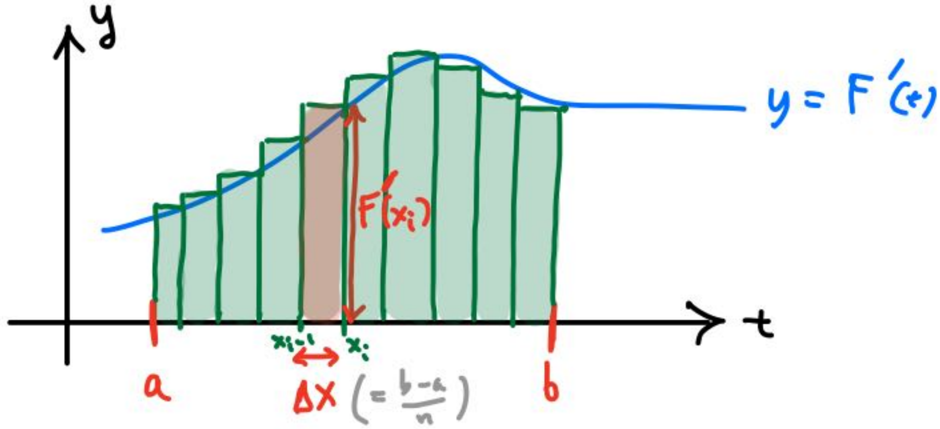
FUNDAMENTAL THEOREM of CALCULUS (v. 2):

$$\int_a^b f(t)\,dt = F(b) - F(a),$$ where F is any antiderivative of f !!

To better understand this, write it as

$$(*) \quad \int_a^b F'(t)\,dt = F(b) - F(a).$$

("integrating rate-of-change over $[a,b]$ gives total change")



The heuristic justification is just that

$$\int_a^b F'(t)\,dt = \text{Area under curve}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} F'(x_i)\,\Delta x \quad \approx \text{change in } F \text{ on } [x_{i-1}, x_i]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(F(x_i) - F(x_{i-1})\right)$$

$$= \lim_{n\to\infty} \Big\{ F(\overset{b}{x_n}) - \cancel{F(x_{n-1}) + F(x_{n-1})} - \cancel{F(x_{n-2}) +} \cancel{\cdots + F(x_2)} - \cancel{F(x_1) + F(x_1)} - F(\underset{a}{x_0}) \Big\}$$

$$= F(b) - F(a).$$

(Once more, the height of F' is the rate of change of F — the higher F' is, → the more F has to change
→ the more area is under F'.)

Ex/ Find $\int_0^1 (x^2 - \sqrt{x})\, dx$.

Use "reverse power rule" to compute the antiderivative:

$$\int_0^1 (x^2 - x^{1/2})\, dx = \left(\frac{x^3}{3} - \frac{x^{3/2}}{3/2}\right)\Big|_0^1 = \left(\frac{1}{3} - \frac{1}{3/2}\right) - (0 - 0)$$

$$= \frac{1}{3} - \frac{2}{3} = -\frac{1}{3}. \quad //$$

TRY: $\int_4^9 \frac{1+\sqrt{x}}{\sqrt{x}}\, dx$ and $\int_0^{\pi} \sin(x)\, dx$

(area under one hump of $\sin(x)$).

Ex/ In some cases finding the antiderivative may take more work:

$$\int_0^{\pi/8} \sec^2(2x)\, dx = \frac{1}{2}\tan(2x)\Big|_0^{\pi/8} = \frac{1}{2}\underbrace{\tan\left(\frac{\pi}{4}\right)}_{1} - \frac{1}{2}\underbrace{\tan(0)}_{0}$$

$$= \frac{1}{2}$$

$$\left(\begin{array}{l} \frac{d}{dx}\tan(x) = \sec^2(x) \\ \frac{d}{dx}\tan(2x) = 2\sec^2(2x) \\ \text{... so divide by 2} \end{array}\right)$$

//

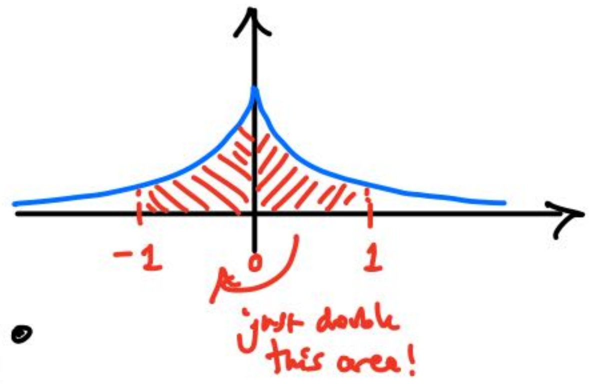
Or in some cases you may have to draw the graph and/or use symmetry properties of the function.

Ex/ $\int_{-1}^{1} e^{-|s|} ds$

$= 2\int_0^1 e^{-s}\, ds$

$= -2e^{-s}\Big|_0^1 = 2e^{-s}\Big|_1^0$

$= 2e^0 - 2e^{-1} = 2(1-e^{-1}).$ //



NOTATION:

- $\int_a^b f(x)\,dx$ is called a <u>definite integral</u> (= number)
- $\int f(x)\,dx$ is called an <u>indefinite integral</u>:

This means "arbitrary antiderivative", i.e.

$$F(x) + \underbrace{C}_{\text{arbitrary constant}},$$

which serves as a reminder that there are an infinite number of functions that are antiderivatives of f (and it doesn't matter which one you choose to compute $\int_a^b$ since all change by the same amount).

Ex/ $\int \sin(x)\,dx = -\cos(x)\,dx + C$

$$\int 1\,dx = x + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| + C \quad \left(\text{not } \frac{x^0}{0} \text{ !!}\right)$$

$$\int e^x\,dx = e^x + C$$

$$\int 3^x\,dx = \frac{3^x}{\ln 3} + C$$

and so on... //

TRY: $\int (x-1)(x+1)\,dx$, $\int \frac{4}{1+x^2}\,dx$,

$$\int \frac{x+3}{x^2-9}\,dx$$

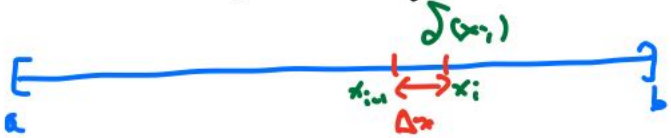
But wait...

WHY DO WE CARE ABOUT COMPUTING #@!%# AREAS!?

Because anytime you need to add stuff up but it doesn't come in discrete packets, you have to integrate — in signal processing, financial math, probability theory/averages, fluid flow (e.g. of blood through a vessel), radioactive decay and chemical reaction kinetics, etc... .

Here's the simplest example that goes beyond "area":

Consider a wire with some electric charge on it. You're given the charge density δ as a function of x



meaning that at x_i, charge is $\delta(x_i)$ per unit length

$\Rightarrow$ total charge in the i^{th} segment of wire is

$$\approx \delta(x_i)\cdot \Delta x\,,$$

and so total charge in the whole wire is

$$\approx \sum_{i=1}^{n} \delta(x_i)\cdot\Delta x \xrightarrow[n\to\infty]{} \int_a^b \delta(x)\,dx.$$

(Seems easy, but what about charge on the surface of a capacitor?)

In general, the integral of any <u>density function</u> gives you the total amount (of whatever).

In physics you have <u>work</u>: amount of force applied times the amount of distance it was applied over: $W = F\cdot D$ provided the force was constant. If the force isn't constant, it's a sum of $\left(\begin{smallmatrix}\text{instantaneous}\\ \text{forces}\end{smallmatrix}\right)\times\left(\begin{smallmatrix}\text{tiny}\\ \Delta x\text{'s}\end{smallmatrix}\right)$:

$$W = \int_a^b F(x)\,dx.$$

Acceleration, velocity, distance

$x(t)$ = distance traveled

$v(t)$ = velocity

$a(t)$ = acceleration

$\frac{dx}{dt} = v$

$\frac{dv}{dt} = a$

So $\int a(t)\,dt = v(t) + B$

$\int v(t)\,dt = x(t) + C$

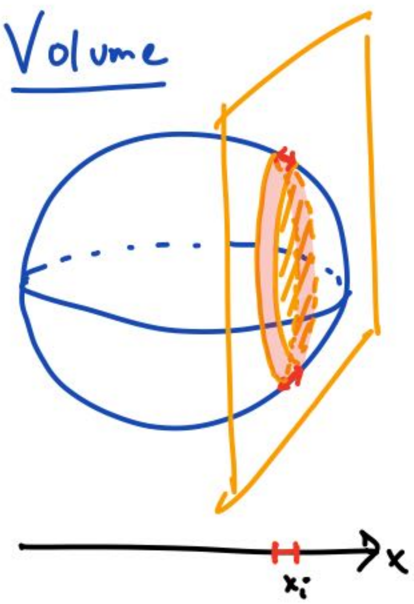
$\int_{t_0}^{t_1} v(t)\,dt = x(t_1) - x(t_0)$ = total distance traveled.

Average velocity $\overline{v}_{[t_0,t_1]} = \frac{\text{total distance traveled}}{\Delta t}$

$= \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} v(t)\,dt$.

Volume

(we'll study this later in the course)



$V_{tot} = \sum (\Delta V)_i$

$= \sum \text{Area}\binom{\text{slice}}{\text{over } x_i} \times \Delta x$

$\Rightarrow$ if know formula for area of slice, get formula for volume of solid!