Lecture 5: How to integrate

Let's review: let $f(x)$ be a function defined on $[a,b]$. $f'(x^*)$ measures the rate-of-change of $f$ at $x^* \in [a,b]$. So $\Delta f \approx f'(x^*) \cdot \Delta x \quad (= \frac{\Delta f}{\Delta x}) \ldots$ BUT:

$f'(x^*) \Delta x$ is also the area of a little box under $f'$.

Adding these up as $x^*$ ranges from $a$ to $b$ gives

$$\sum_{i=1}^{n} f'(x_i^*) \Delta x \approx \lim_{n \to \infty} \sum_{i=1}^{n} (\Delta f)_i = \text{total change in } f$$

$$\int_{a}^{b} f'(x) \, dx \quad \Rightarrow \quad f(b) - f(a)$$

$$\Rightarrow \text{area under derivative } f' = \text{change in } f \ .$$

$$\Rightarrow \text{area under } f = \text{change in antiderivative} \ !$$

Let's approach this a bit differently:

Given $f$, let $F$ be any antiderivative. e.g.

- $f(x) = x^n \quad \Rightarrow \quad F(x) = \frac{x^{n+1}}{n+1}$
- $f(x) = \sin(x) \quad \Rightarrow \quad F(x) = -\cos(x)$
Now recall

\[ \text{FTC v.1.0} \quad \frac{d}{dx} \int_a^x f(t) \, dt = f(x) ; \text{ that is,} \]

\[ f(x) = \int_a^x f(t) \, dt \text{ is an antiderivative of } f(x) . \]

So \( f' = f = F' \) \( \Rightarrow (F_x - F)' = f - f = 0 \)

\( \Rightarrow \quad F_x - F = C \quad (\text{constant}) \)

\( \Rightarrow \quad \int_a^x f(t) \, dt = F(x) + C \)

\( \Rightarrow \quad 0 = \int_a^a f(t) \, dt = F(a) + C \quad (\text{so } C = -F(a)) \)

\( \Rightarrow \quad \int_a^b f(t) \, dt = F(b) - F(a) = \left. F(x) \right|_a^b \),

confirming that we can compute definite integrals just by finding the change in the antiderivative:

e.g. \( \int_a^b x^n \, dx = \left. \frac{x^{n+1}}{n+1} \right|_a^b = \frac{b^{n+1} - a^{n+1}}{n+1} \).

**Fundamental Theorem of Calculus (v.2):**

\[ \int_a^b f(t) \, dt = F(b) - F(a) , \quad \text{where } F \text{ is any antiderivative of } f . \]
To better understand this, write it as

\[(a) \int_a^b F'(t) \, dt = F(b) - F(a) \quad \text{("integrating rate-of-change over \([a,b]\) gives total change")}
\]

The heuristic justification is just that

\[
\int_a^b F'(t) \, dt = \text{Area under curve} \\
= \lim_{n \to \infty} \sum_{i=1}^{n} F'(x_i) \Delta x \\
= \lim_{n \to \infty} \sum_{i=1}^{n} \left( F(x_i) - F(x_{i-1}) \right) \\
= \lim_{n \to \infty} \left\{ F(x_1) - F(x_0) + F(x_2) - F(x_1) + \ldots + F(x_n) - F(x_{n-1}) \right\} \\
= F(b) - F(a) .
\]

(Once more, the height of \( F' \) is the rate of change of \( F \) — the higher \( F' \) is, < the more \( F \) has to change; the more area is under \( F' \).)
Ex/ Find \( \int_0^1 (x^2 - \sqrt{x}) \, dx \).

Use “reverse power rule” to compute the antiderivative:
\[
\int_0^1 (x^2 - x^{1/2}) \, dx = \left[ \frac{x^3}{3} - \frac{x^{3/2}}{3/2} \right]_0^1 = \left( \frac{1}{3} - \frac{1}{3/2} \right) - (0 - 0) = \frac{1}{3} - \frac{2}{3} = -\frac{1}{3}.
\]

\[\text{TRY: } \int_4^9 \frac{1 + \sqrt{x}}{\sqrt{x}} \, dx \text{ and } \int_0^\pi \sin(x) \, dx\]

(area under one hump of \( \sin(x) \)).

Ex/ In some cases finding the antiderivative may take more work:
\[
\int_0^{\pi/4} \sec^2(2x) \, dx = \left[ \frac{1}{2} \tan(2x) \right]_0^{\pi/4} = \frac{1}{2} \tan\left(\frac{\pi}{4}\right) - \frac{1}{2} \tan(0) = \frac{1}{2}
\]

\[\frac{d}{dx} \tan(x) = \sec^2(x)\]
\[\frac{d}{dx} \tan(2x) = 2 \sec^2(2x)\]
...so divide by 2

Or in some cases you may have to draw the graph and/or use symmetry properties of the function.
Ex: \[ \int_{-1}^{1} e^{-1s} \, ds \]

\[= 2 \int_{0}^{1} e^{-s} \, ds \]

\[= -2 e^{-s} \bigg|_{0}^{1} = 2 e^{-s} \bigg|_{0}^{1} \]

\[= 2e^0 - 2e^{-1} = 2(1-e^{-1}) \].

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**NOTATION:**

- \( \int_{a}^{b} f(x) \, dx \) is called a definite integral (= number)

- \( \int f(x) \, dx \) is called an indefinite integral:
  
  This means “arbitrary antiderivative”, i.e.

  \[ F(x) + C \]

  which serves as a reminder that there are an infinite number of functions that are antiderivatives of \( f \) (and it doesn’t matter which one you choose to compute \( \int_{a}^{b} \), since all change by the same amount).
Ex/ \begin{align*}
    \int \sin(x) \, dx &= -\cos(x) + C \\
    \int 1 \, dx &= x + C \\
    \int x^n \, dx &= \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1 \\
    \int \frac{1}{x} \, dx &= \ln|x| + C \quad \text{(not } \frac{x^0}{0} \text{!)} \\
    \int e^x \, dx &= e^x + C \\
    \int 3^x \, dx &= \frac{3^x}{\ln 3} + C \\
    \text{and so on...} \\
\end{align*}

TRY: \int (x-1)(x+1) \, dx, \int \frac{4}{1+x^2} \, dx, \int \frac{x+3}{x^2-9} \, dx

BUT WAIT...

WHY DO WE CARE ABOUT COMPUTING #R# AREAS #?#?

Because anytime you need to add stuff up but it doesn't come in discrete packets, you have to integrate — in signal processing, financial math, probability theory/averages, fluid flow (e.g. of blood through a vessel), radioactive decay and chemical reaction kinetics, etc...
Here's the simplest example that goes beyond "area": consider a wire with some electric charge on it. You're given the charge density $\delta$ as a function of $x$:

\[
\begin{array}{c}
\text{\textbullet} \\
\text{a} \\
\text{x}_{i-1} \quad \text{x}_i \quad \text{x}_{i+1} \\
\text{b}
\end{array}
\]

meaning that at $x_i$, charge is $\delta(x_i)$ per unit length.

Total charge in the $i^\text{th}$ segment of wire is

\[\approx \delta(x_i) \cdot \Delta x,\]

and total charge in the whole wire is

\[\approx \sum_{i=1}^{n} \delta(x_i) \cdot \Delta x \quad \longrightarrow \quad \int_{a}^{b} \delta(x) \, dx.\]

(Seems easy, but what about charge on the surface of a capacitor?)

In general, the integral of any density function gives you the total amount (of whatever).

In physics you have work: amount of force applied times the amount of distance it was applied over:

\[W = F \cdot D\] provided the force was constant. If the force isn't constant, it's a sum of (instantaneous) forces ($\delta x$):

\[W = \int_{a}^{b} F(x) \, dx.\]
**Acceleration, Velocity, Distance**

\[ x(t) = \text{distance traveled} \]
\[ v(t) = \text{velocity} \]
\[ a(t) = \text{acceleration} \]

\[ \frac{dx}{dt} = v \]
\[ \frac{dv}{dt} = a \]

so

\[ \int a(t) \, dt = v(t) + B \]
\[ \int v(t) \, dt = x(t) + C \]
\[ \int_{t_0}^{t_1} v(t) \, dt = x(t_1) - x(t_0) = \text{total distance traveled.} \]

Average velocity \( \bar{v}(t_i, t_f) = \frac{\text{total distance traveled}}{\Delta t} \)

\[ = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} v(t) \, dt . \]

**Volume**

(we’ll study this later in the course)

\[ V_{\text{tot}} = \sum (\Delta V)_i \]
\[ = \sum \text{Area (slice over } x_i) \times \Delta x \]

\[ \Rightarrow \text{If know formula for area of slice, get formula for volume of solid!} \]