Lecture 6: Exam I Review

As mentioned last week in class, the first exam will comprise 10 multiple-choice problems and two 2-part hand-graded problems (2 pp. of written work). It covers §§ 4.9, 5.1, & 5.2 and is worth 10% of your grade.

Let's take stock of what we have covered so far:

- <u>Antiderivatives</u>: given f, F is an antiderivative of f if $F' = f$. (Any 2 antiderivatives differ by a constant.) Which ones should you know?

  x^n (incl. x^{-1}), a^x, $\sin$ & $\cos$, $\sec^2$ & $\sec \cdot \tan$, $\frac{1}{1+x^2}$ & $\frac{1}{\sqrt{1-x^2}}$, and variants where x is replaced by Cx.

- <u>Riemann sums</u>: given f on $[a,b]$, how to write a sum of areas of n rectangles approximating the "signed area"  $A_+ - A_-$: subdivide $[a,b]$ into n intervals of equal length $\Delta x = \frac{b-a}{n}$, given by $[x_{i-1}, x_i]$ (where $x_i = a + i\Delta x$), with sample points $x_i^* \in [x_{i-1}, x_i]$, and write $R_n = \Delta x \sum_{i=1}^{n} f(x_i^*)$.

Know what upper/lower sums, left/right sums, midpoint rule are, and that $\lim_{n\to\infty} \mathcal{R}_n$ computes the . . .

- Definite integral: $\int_a^b f(x)\,dx$. Be able to write the limit of Riemann sums that calculates the integral, and be able to do this computation in the simplest cases (e.g. if all you need to use is $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$). Be able to recognize what integral a given "$\lim_{n\to\infty} \mathcal{R}_n$" is computing. Know that the integral computes the signed area, and be able to use what you know about areas (circles, rectangles, triangles) to compute easy ones.

- Properties of the definite integral: $\int_a^a = 0$; linearity ; $\int_a^b = -\int_b^a$; $\int_a^b + \int_b^c = \int_a^c$; comparison theorem ; and $m \le f(x) \le M \Rightarrow m(b-a) \le \int_a^b f\,dx \le M(b-a)$.

Now we turn to problems.

Antiderivatives

(1) If $f''(x) = -9\sin 3x$, $f'(0) = 0$, and $f(0) = 2$, find $f(\pi/4)$.

$$f'(x) = 3\cos 3x + C_0 \Rightarrow C_0 = -3$$
$$f(x) = \sin 3x - 3x + C_1 \Rightarrow C_1 = 2$$
$$\Rightarrow f(x) = \sin(3x) - 3x + 2$$

(2) Find an antiderivative of $\dfrac{3x(x+2)}{1+x^2}$.

$$\frac{3x^2+6x}{1+x^2} = 3\cdot\frac{2x}{1+x^2} + 3\,\cancel{\frac{1+x^2}{1+x^2}}^{\,1} - 3\,\frac{1}{1+x^2}$$

$$\xrightarrow{\text{ANTI}} 3\ln(1+x^2) + 3x - 3\arctan(x)$$

Riemann sums and integrals

(3) Write $\int_2^{10} x^6\,dx$ as a limit of (right-handed) Riemann sums.

$\lim_{n\to\infty} \frac{8}{n} \sum_{i=1}^{n} \left(2+\frac{8i}{n}\right)^6$

(4) What integral does $\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}}+\sqrt{\frac{2}{n}}+\cdots+\sqrt{\frac{n}{n}}\right)$ compute?
(So, what is this limit?)

$\int_0^1 \sqrt{x}\,dx = \frac{x^{3/2}}{3/2}\Big|_0^1 = \frac{2}{3}$

(5) Consider the table

x	0	1	2	3	4	5	6	7	8
$f(x)$	-1	3	1	2	2	1	-1	-1	1

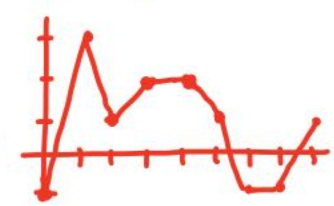
Assume that the function takes a straight line path between its values at each integer.
Find the upper, lower, right, left, and midpoint Riemann sums with $n=4$ on $[0,8]$.

$U=16,\ L=-4,\ R=6,\ L=2,\ M=10$

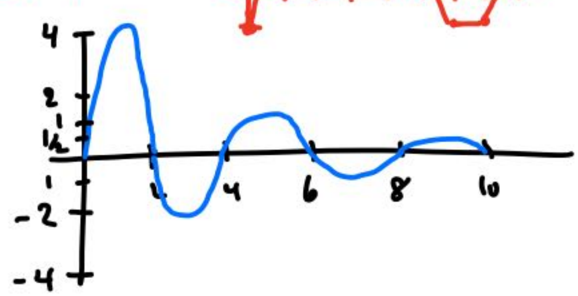
Integrals and areas

(6) Find $\int_0^8 f(x)\,dx$ in (5)

it's 7: use



(7) Let $g(x) = \int_0^x f(t)\,dt$ where f is as depicted.



which are true?

(a) g attains an absolute max at $x=2$

(b) g has a local max at $x=5$

(c) g has a local min at $x=4$

(d) g is concave down on $[0,2]$

(a) & (c)

(8) What are the best possible upper & lower bounds U & L that one can obtain on $\int_0^{10} f(x)\,dx$ in (7) using 5 rectangles of equal width?

$U = 11$
$L = -6$

Properties of the Integral

(9) Which are true? (a) only

(a) f continuous with minimum of 3 on $[2,4] \Rightarrow$
$$\int_2^4 f(x)\,dx \geq 6$$

(b) $\int_1^4 g(x)\,dx = -5$ and $\int_4^3 g(x)\,dx = 2 \Rightarrow \int_3^1 g(x)\,dx = -7$

(10) Suppose you know $\int_0^b f(x)\,dx = \ln(b+1)$ for $b > 0$.
What is $\int_3^5 (3f(x) - 2)\,dx$?

$$= 3\int_3^5 f(x)\,dx - 2\int_3^5 dx$$

$$= 3\int_0^5 f(x)\,dx - 3\int_0^3 f(x)\,dx - 2\cdot 2$$

$$= 3\ln(5+1) - 3\ln(3+1) - 4$$

$$= 3\ln(3/2) - 4$$