Lecture 7: More on integrals

also, review of EXAM 1

For any differentiable function F on $[a,b]$,

- $$\underbrace{\int_a^b F'(x)\,dx}_{\text{(definite) integral of rate of change}} = F(x)\Big|_a^b := \underbrace{F(b)-F(a)}_{\text{"net change"}} \qquad \text{(FTC v.2)}$$

- $$\underbrace{\int F'(x)\,dx}_{\text{indefinite integral}} = F(x) + C.$$

Let's look at some calculations and interpretations.

Ex/ $\displaystyle\int_0^{\pi/3} \frac{\sin\theta + \sin\theta\tan^2\theta}{\sec^2\theta}\,d\theta = ?$

Pythagorean theorem: $\cos^2\theta + \sin^2\theta = 1 \underset{/\cos^2\theta}{\Longrightarrow} 1+\tan^2\theta = \sec^2\theta$

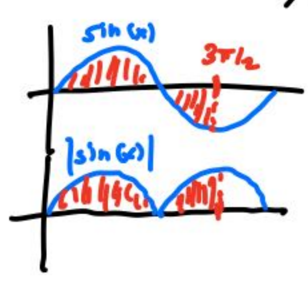
so integral is just $\displaystyle\int_0^{\pi/3} \sin\theta\,d\theta = -\cos\theta\Big|_0^{\pi/3}$

$\displaystyle = \cos\theta\Big|_{\pi/3}^{0} = \cos 0 - \cos\pi/3 = 1 - \tfrac{1}{2} = \tfrac{1}{2}.$

Ex/ $\displaystyle\int_{-1}^{1} x e^{-x^4}\,dx = ?$

It's zero by symmetry (you can't compute the anti-derivative!).

Ex/ $\displaystyle\int_0^{3\pi/2} |\sin x|\,dx = ?$ Draw the graph:



$\displaystyle\int_0^{\pi} \sin x\,dx + \int_{\pi}^{3\pi/2} (-\sin x)\,dx = 3.$

Ex/ $\int \frac{\sin 2x}{\sin x}\,dx = \int \frac{2\cancel{\sin x}\cos x}{\cancel{\sin x}}\,dx = \int 2\cos x\,dx = 2\sin x + C.$

Ex/ $\int \sqrt{\frac{1+x^2}{1-x^4}}\,dx = \int \sqrt{\frac{\cancel{1+x^2}}{\cancel{(1+x^2)}(1-x^2)}}\,dx = \int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C.$
(i.e. $\sin^{-1} x$)

Ex/ $\int \frac{dx}{\sqrt[5]{x}} = \int x^{-1/5}\,dx = \frac{x^{4/5}}{4/5} + C = \frac{5}{4}x^{4/5} + C$

Ex/ I toss a ball up at speed $v_0 = 20\text{ft}/s$, while standing of the edge of a 20 ft. building. How many seconds until it hits the ground? What was the cumulative change in height of the ball from $t=0$ to $t=2$ seconds?

- $a = -32\ \text{ft}/s^2$ (constant accel. due to gravity)
  $= \frac{dv}{dt} \Rightarrow \int \frac{dv}{dt}\,dt = \int a\,dt$
  $\Rightarrow v(t) = at + C = -32t + C$ ← clearly v_0
  $= -32t + 20$
- $v = \frac{dh}{dt} \Rightarrow h(t) = \int \frac{dh}{dt}\,dt = \int v\,dt = -16t^2 + 20t + K$ ← clearly h_0
  $= -16t^2 + 20t + 50$
- Now set $0 = h(t_1) = -16t_1^2 + 20t_1 + 50$ & solve
  ↑ time hits ground
  $= -16\left(t_1 - \frac{5}{2}\right)\left(t_1 + \frac{5}{4}\right)$
  $\Rightarrow t_1 = 5/2 = 2.5\,s$ ← $t_1 = -\frac{5}{4}$ not an option!
- Finally, $h(2) - h(0) = \int_0^2 h'(t)\,dt = \int_0^2 v(t)\,dt = \int_0^2 (-32t + 20)\,dt$
  $= \left[-16t^2 + 20t\right]_0^2 = -16\cdot 2^2 + 20\cdot 2 = -64 + 40 = -24$ ft.

I mentioned a grab-bag of applications at the end of lecture 5. Let's try a couple.

Ex/ The mass density of a 5-cm piece of wire is given by $\delta(x) = 2 + \cos^2(\pi x)$ (in mg/cm). What is the total mass?

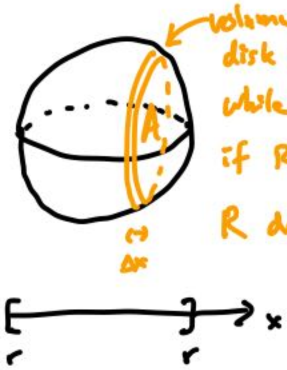
$$\text{mass} = \int_0^5 \left(2 + \underbrace{\cos^2(\pi x)}\right) dx = \int_0^5 \left(\frac{5}{2} + \frac{1}{2}\cos(2\pi x)\right) dx$$

you may recall that $\cos 2t = \cos^2 t - \sin^2 t \underset{\uparrow\ \text{Pythag. thm.}}{=} 2\cos^2 t - 1$

$\Rightarrow \cos^2 t = \frac{\cos 2t + 1}{2}$

$$= \left[\frac{5}{2}x + \frac{1}{4\pi}\sin(2\pi x)\right]_0^5 = \frac{25}{2} \text{ mg}.$$

Ex/ What is the volume of a sphere of radius r?



volume of a little disk is $\Delta x \cdot A$, while $A = \pi R^2$ if R = radius of disk.

R depends on x by $R(x) = \sqrt{r^2 - x^2}$

$$V = \int_{-r}^{r} \pi \left(\sqrt{r^2 - x^2}\right)^2 dx$$

$$= \pi \int_{-r}^{r} (r^2 - x^2)\, dx$$

$$= \pi \left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r}$$

$$= \pi \left\{ \underbrace{\left(r^3 - \frac{r^3}{3}\right)}_{\frac{2}{3}r^3} - \underbrace{\left(-r^3 + \frac{r^3}{3}\right)}_{-\frac{2}{3}r^3} \right\}$$

$$= \frac{4}{3}\pi r^3.$$

(What about a hypersphere?)

We'll be treating volumes more thoroughly later in the course.

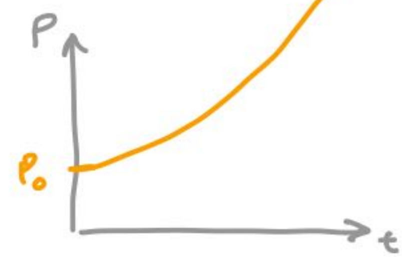
Here's a more exotic application (meaning that it's not something we do much of in this course):

just for entertainment

Ex/ (population growth) Let $P(t)$ = population at time t.

If there are no constraints on growth, then P' is proportional to P:

$$\frac{dP}{dt} = \alpha P$$



$$\frac{1}{P}\,dP = \alpha\,dt$$

$$\int \frac{dP}{P} = \int \alpha\,dt$$

$$\ln P = \alpha t + C$$

raise both sides to e

$$P = e^{\alpha t + C} = P_0 e^{\alpha t} \Rightarrow$$ population growth is exponential.

(Same thing applies to compounding interest.)

What if there's a constraint? i.e., if the population rises above M, it starts to decrease:

$$\frac{dP}{dt} = \alpha P\cdot\left(1 - \frac{P}{M}\right)$$ Yikes!

assume P_0 is less than M

$$\frac{M\,dP}{P(M-P)} = \alpha\,dt$$

$$\int \frac{M\,dP}{P(M-P)} = \int \alpha\,dt$$

you'll learn how to do integrals like this when we study partial fractions

$$\frac{d}{dP}\ln\left(\frac{P}{M-P}\right) = \frac{d}{dP}\left(\ln(P) - \ln(M-P)\right) = \frac{1}{P} + \frac{1}{M-P} = \frac{M-P+P}{P(M-P)} = \frac{M}{P(M-P)}.$$

$$\ln\left(\frac{P}{M-P}\right) = \alpha t + C$$

$e^{-(\dots)}$

$$\frac{M}{P} - 1 = \frac{M-P}{P} = Ke^{-\alpha t} \quad \left(\Rightarrow K = \frac{M}{P_0} - 1\right)$$

$$\frac{M}{P} = 1 + Ke^{-\alpha t}$$

$$P(t) = \frac{M}{1 + \left(\frac{M}{P_0} - 1\right)e^{-\alpha t}}$$