Lecture 8 : Substitution (or the "Backwards Chain Rule")

Today we'll see how to compute integrals when your star player x just isn't performing — pull him out and make a u-substitution!

WARM-UP: (a) $\displaystyle\int_0^{\pi/6} \cos(3x)\,dx = ?$

(b) $\displaystyle\int_0^1 \frac{dx}{(1+\sqrt{x})^4} = ??!$ (options: $0, \frac{1}{6}, \frac{1}{3}, 1, 3, 6$)

Ex 1/ Consider the problem of finding

$$\int (2x+3)\cos(x^2+3x)\,dx.$$

By the Chain Rule,

$$\frac{d}{dx}\underbrace{\sin(x^2+3x)}_{f(g(x))} = \underbrace{\left(\frac{d}{dx}(x^2+3x)\right)}_{g'(x)}\cdot\underbrace{\cos(x^2+3x)}_{f'(g(x))}$$

$$= (2x+3)\cos(x^2+3x),$$

and so $\underline{\sin(x^2+3x)+C}$ is the answer. But it's not always going to be easy to eyeball the integral like this, so let's do it using a safer method:

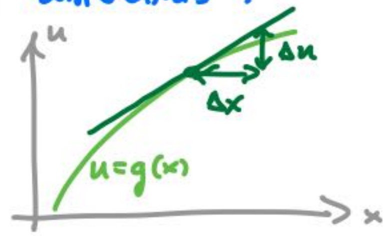
- let $u = x^2+3x$
- then $\dfrac{du}{dx} = 2x+3 \rightsquigarrow du = (2x+3)\,dx.$

<u>ASIDE</u>: This might seem questionable since $\frac{du}{dx}$ isn't an actual fraction. But it's OK to <u>think</u> about du & dx as "infinitesimally small" Δu & Δx, called "differentials".

Remember the approximation formula

$$u = g(x) \rightsquigarrow \Delta u = \Delta g \approx g'(x)\,\Delta x$$

$$\xrightarrow{\text{version with differentials}} du = g'(x)\,dx.$$



So let's substitute: with u, du as above

- $\int (2x+3)\cos(x^2+3x)\,dx = \int \cos(u)\,du = \sin(u) + C$

but you're not (quite) done, unless you like to make a habit of answering questions in a different language. (While impressive, that would be a little strange.) In this case, we haven't answered the question in terms of x

- substitute back in $\rightsquigarrow = \sin(x^2+3x) + C$.

(<u>Now</u> we're done.) So what is the general rule behind this?

<u>(INDEFINITE) SUBSTITUTION RULE</u>: Let F be an antiderivative of f, and let $u = g(x)$. Then

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du = F(u) + C$$

$$= F(g(x)) + C$$

<u>Check</u>: $\frac{d}{dx}F(g(x)) = F'(g(x))\,g'(x) = f(g(x))\,g'(x)$ (by the Chain Rule).

While the example was easy, it can be tricky <u>in practice</u> choosing u (i.e., what $g(x)$ should be).

Ex 2/ $\displaystyle\int \frac{4x^3}{1+x^4}\,dx = \int \frac{du}{u} = \ln|u| + C$
$= \ln|1+x^4| + C.$

Think: the denominator's derivative is exactly the numerator.
So take $u = 1+x^4 \rightsquigarrow du = 4x^3\,dx$

Can we find something this is derivative of

Ex 3/ $\displaystyle\int \frac{4x^3}{1+2x^4+x^8}\,dx = \int \frac{4x^3}{(1+x^4)^2}\,dx$

so again...
$u = 1+x^4$
$du = 4x^3$

$\displaystyle\rightsquigarrow = \int \frac{du}{u^2} = -\frac{1}{u} + C = \frac{-1}{1+x^4} + C.$

again: is an antiderivative of this lurking in the denominator?

Ex 4/ $\displaystyle\int \frac{2x}{1+x^4}\,dx = \int \frac{2x}{1+(x^2)^2}\,dx = \int \frac{du}{1+u^2}$

$u = x^2$
$du = 2x\,dx$

$= \arctan(u) + C$
$= \arctan(x^2) + C.$

<u>ASIDE</u>: Example 2 illustrates the useful general rule

$$\int \frac{g'(x)}{g(x)}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|g(x)| + C.$$

(with $du = g'(x)\,dx$ and $u = g(x)$)

e.g. $\displaystyle\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = \int \frac{-du}{u} = -\ln|u| + C$

(with $-du = \sin(x)\,dx$ and $u = \cos(x)$)

$\displaystyle = -\ln|\cos(x)| + C = \ln\left|\frac{1}{\cos(x)}\right| + C = \ln|\sec(x)| + C.$

(Similarly one gets $\int \cot(x)\,dx = \ln|\sin(x)| + C$.)

Another one is

$$\int g'(x)\,e^{g(x)}\,dx = e^{g(x)} + C,$$

TRY: $\int x^2 e^{x^3}\,dx = \,??\, + C$

$e^{\frac{1}{3}x^3}$

$\frac{1}{3}x^3 e^{x^3}$

e^{x^3}

$\frac{1}{3}e^{x^3}$

You'd think, with u-substitution just being the "Chain rule backwards", you could use it in definite integrals too:

$$(*)\quad \int_0^3 x\sqrt{x^2+1}\,dx \;=\; \int_0^3 \sqrt{u}\cdot\tfrac{1}{2}\,du \quad\text{, right ?}$$

$u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}\,du = x\,dx$

WRONG!

Problem: we changed to an integral in u, but as x goes from 0 to 3, u __doesn't__ go from 0 to 3!

$x = 0 \rightsquigarrow u = 0^2+1 = 1$

$x = 3 \rightsquigarrow u = 3^2+1 = 10$

$$\text{So } (*) = \int_1^{10} \sqrt{u}\cdot\tfrac{1}{2}\,du = \tfrac{1}{2}\int_1^{10} u^{1/2}\,du$$

$$= \frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right]_1^{10} = \frac{1}{\cancel{2}}\left(\frac{10^{3/2}}{3/\cancel{2}} - \frac{1^{3/2}}{3/\cancel{2}}\right)$$

$$= \frac{1}{3}\left(10^{3/2} - 1\right).$$

So changing the limits IS annoying, but at least you don't then have to substitute $g(x)$ back in for u, right?

(DEFINITE)
SUBSTITUTION RULE: Let F be an antiderivative of f, and let $u = g(x)$. Then

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du = \Big[F(u)\Big]_{g(a)}^{g(b)}$$

$$\Big(= F(g(b)) - F(g(a)) \Big).$$

Ex 5/

$$\int_0^{\pi/3} \sin(x)\cos(x)\,dx = \int_0^{\sqrt{3}/2} u\,du = \left[\frac{u^2}{2}\right]_0^{\sqrt{3}/2}$$

$$\begin{bmatrix} u = \sin(x) \\ du = \cos(x)\,dx \\ x = 0 \rightsquigarrow u = \sin(0) = 0 \\ x = \pi/3 \rightsquigarrow u = \sin(\pi/3) = \sqrt{3}/2 \end{bmatrix}$$

$$= \frac{3/4}{2} - \frac{0}{2}$$

$$= \frac{3}{8}.$$

Ex 6/

$$\int_5^9 \frac{x}{x-3}\,dx = \int_5^9 \left(\frac{x-3+3}{x-3}\right) dx$$

$$= \int_5^9 \left(\frac{x-3}{x-3} + \frac{3}{x-3}\right) dx = \int_5^9 \left(1 + \frac{3}{x-3}\right) dx$$

$$= \int_2^6 \left(1 + \frac{3}{u}\right) du = \Big[u + 3\ln|u|\Big]_2^6$$

$$\begin{bmatrix} u = x-3 \\ du = dx \\ x = 5 \rightsquigarrow u = 2 \\ x = 9 \rightsquigarrow u = 6 \end{bmatrix}$$

$$= (6 + 3\ln 6) - (2 + 3\ln 2)$$

$$= 4 + 3(\ln 6 - \ln 2)$$

$$= 4 + 3\ln(6/2)$$

$$= 4 + 3\ln 3.$$

Finally, let's come back to

$$\boxed{\int_0^1 \frac{dx}{(1+\sqrt{x})^4} = \begin{matrix} 0 \\ 1/6 \\ 1/3 \\ 1 \\ 3 \\ 6 \end{matrix} \quad ?}$$

Should be easier now (what is u?).

P.S. I should mention the

EVEN/ODD RULE

- f even $[f(-x) = f(x)] \Longrightarrow$
$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$
- f odd $[f(x) = -f(x)] \Longrightarrow$
$$\int_{-a}^{a} f(x)\,dx = 0$$

which is seen by breaking the integral in two $\left(\int_0^a + \int_{-a}^0\right)$ and substituting $u = -x$ ($du = -dx$) to get

$$\int_{-a}^0 f(x)\,dx = \int_a^0 f(-u)\,(-du) = -\int_a^0 f(-u)\,du$$
$$= \int_0^a f(-u)\,du = \begin{cases} \int_0^a f(u)\,du, & f \text{ even} \\ -\int_0^a f(u)\,du, & f \text{ odd}. \end{cases}$$