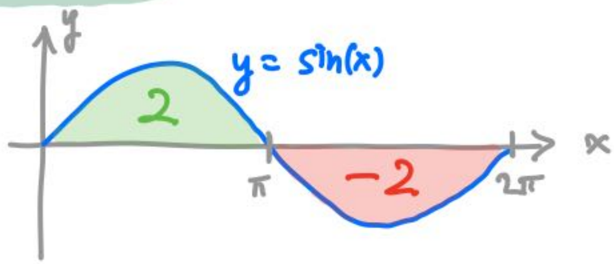
Lecture 9 : Areas between curves

Pretty much by definition, definite integrals compute (signed) areas "under" curves:

Try

$$\int_0^{\pi} \sin(x)\,dx = \Big[-\cos(x)\Big]_0^{\pi} = 2$$

$$\int_0^{2\pi} \sin(x)\,dx = \Big[-\cos(x)\Big]_0^{2\pi} = 0$$



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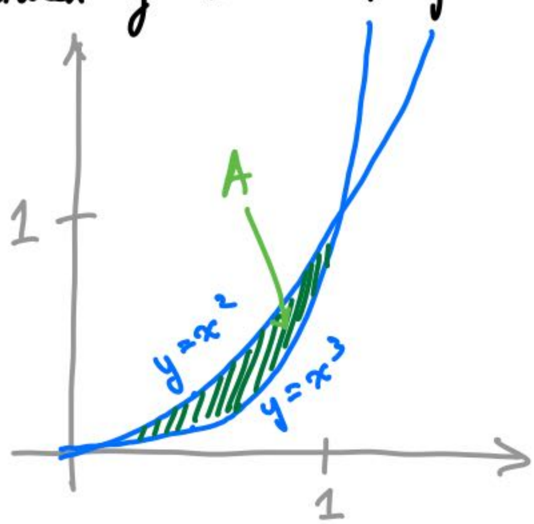
Now here's some true excitement: let's find the area between $y = x^2$ and $y = x^3$. Looks like they intersect at (0,0) and (1,1) with a "crescent moon" region in between. Check:



$$x^2 = x^3$$
$$x^2 - x^3 = 0$$
$$x^2(1-x) = 0$$
$$x = 0 \text{ or } 1$$
$$\Downarrow \qquad \Downarrow$$
$$y = 0 \qquad y = 1.$$

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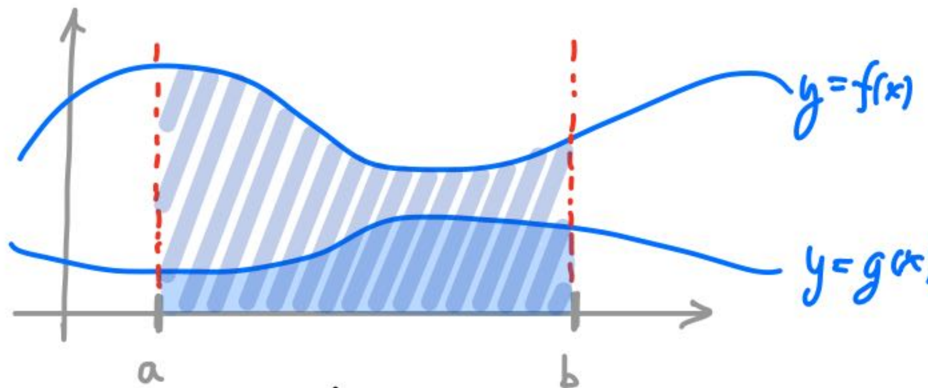
The second thing to note is that for $x \in [0,1]$,

$$x^2 \geq x^3.$$

Now let's think about the area between two curves



$$\text{Area under } f(x) = \int_a^b f(x)\,dx$$

$$\qquad | \qquad\qquad\qquad | $$

$$\text{Area under } g(x) = \int_a^b g(x)\,dx$$

$$\qquad \| \qquad\qquad\qquad \| $$

$$\text{Area in between} = \int_a^b (f(x) - g(x))\,dx \qquad (*)$$

Remark: if x = time and $y = f(x), g(x)$ = speed of two vehicles, the area between the curves is the difference of the distances they travelled.

So $A = \int_0^1 (x^2 - x^3)\,dx$

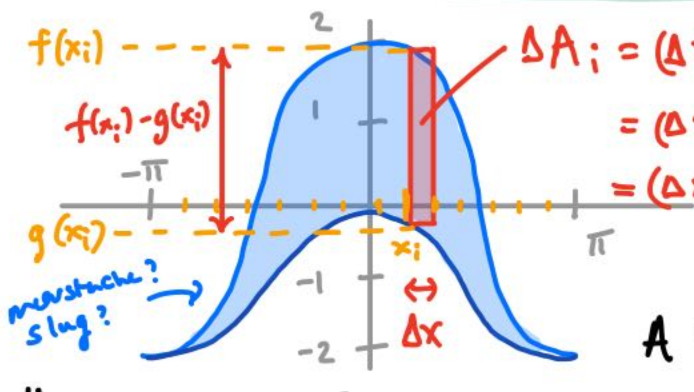
$$= \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \left[\frac{1^3}{3} - \frac{1^4}{4}\right] - [0 - 0]$$

$$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

There's a way to think about (*) that generalizes better to other situations:

1. Sketch the region;
2. Slice it into thin pieces (strips);
   (& label a particular piece)
3. Approximate the area of this typical piece, pretending it is a rectangle;
4. Add up these approximations; an
5. Take the limit as the width of the pieces $\to 0$, thus obtaining a definite integral.

Try Find the area between $y = 2\cos x$ $(= f(x))$ and $y = \cos(x) - 1$ $(= g(x))$.

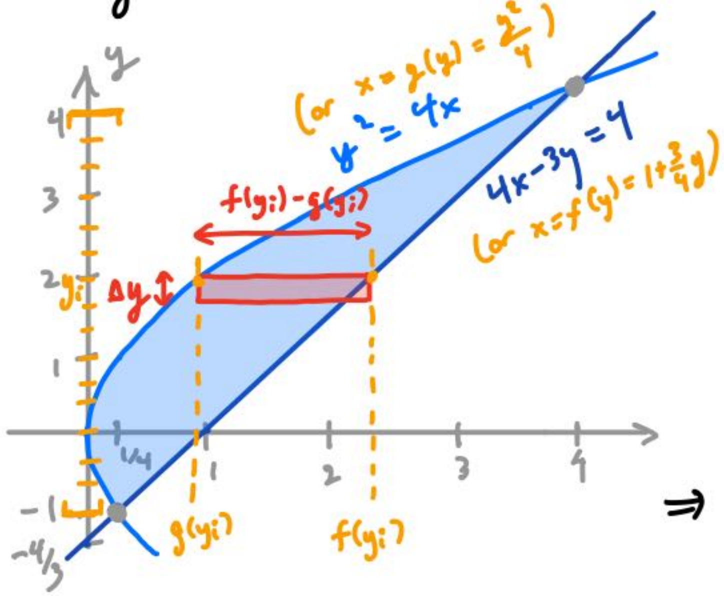


$$\Delta A_i = (\Delta x)(f(x_i) - g(x_i))$$
$$= (\Delta x)(2\cos(x_i) - \cos(x_i) + 1)$$
$$= (\Delta x)(\cos(x_i) + 1)$$
$$\Downarrow$$
$$A \approx \sum_{i=1}^{n} \Delta A_i = \Delta x \sum_{i=1}^{n} (\cos(x_i) + 1)$$

$$\xRightarrow{\lim_{n\to\infty}} A = \int_{-\pi}^{\pi} (\cos(x) + 1)\, dx = \Big[\sin(x) + x\Big]_{-\pi}^{\pi}$$

$$= \left[\cancel{\sin(\pi)}^{0} + \pi\right] - \left[\cancel{\sin(-\pi)}^{0} - \pi\right] = \pi - (-\pi) = 2\pi. \quad //$$

Ex/ Find the area of the region between the parabola $y^2 = 4x$ and the line $4x - 3y = 4$.



To find the points of intersection:

$$4x - 3y = 4$$
$$4x = 4 + 3y$$
$$\parallel$$
$$y^2 \text{ (from } 1^{st} \text{ eqn.)}$$
$$\Rightarrow y^2 - 3y - 4 = 0$$
$$(y-4)(y+1) = 0.$$

Imagine now slicing this region vertically. You face a problem, because the lower boundary consists of two different curves. This means using one integral to find the area from $x = 0$ to $x = \frac{1}{4}$, and another to get the rest: writing the equations of the curves as

$$y = \pm\sqrt{4x} \quad \text{and} \quad y = \frac{4}{3}(x-1),$$

$$A = \int_0^{1/4} \left\{ \sqrt{4x} - \left(-\sqrt{4x}\right) \right\} dx + \int_{1/4}^{4} \left\{ \sqrt{4x} - \frac{4}{3}(x-1) \right\} dx$$

$$= 4\int_0^{1/4} \sqrt{x}\, dx + \int_{1/4}^{4} \left(2\sqrt{x} - \frac{4}{3}x + \frac{4}{3} \right) dx$$

$$= 4\left[\frac{x^{3/2}}{3/2} \right]_0^{1/4} + \left[2\frac{x^{3/2}}{3/2} - \frac{\cancel{4}^{\,2}}{3}\frac{x^2}{\cancel{2}} + \frac{4}{3}x \right]_{1/4}^{4}$$

$$= 8\frac{(1/4)^{3/2}}{3} + \left(\frac{4}{3}\cdot 4^{3/2} - \frac{2}{3}4^2 + \frac{4}{3}\cdot 4 \right) - \left(\frac{4}{3}\cdot\left(\frac{1}{4}\right)^{3/2} - \frac{2}{3}\left(\frac{1}{4}\right)^2 + \frac{4}{3}\cdot\frac{1}{4} \right)$$

$4^{3/2} = 8$

$\left(\frac{1}{4}\right)^{3/2} = \frac{1}{8}$

$$= \frac{1}{3} + \left(\cancel{\frac{32}{3}} - \cancel{\frac{32}{3}} + \frac{16}{3}\right) - \left(\frac{1}{6} - \frac{1}{24} + \frac{1}{3}\right) = \frac{17}{3} - \frac{11}{24} = \frac{125}{24},$$

which was a mess.

Of course, it's silly to do it this way: it's much easier to slice horizontally. Writing the curves' equations

$$x = \frac{1}{4}y^2 \quad \text{and} \quad x = 1 + \frac{3}{4}y,$$

$$\Delta A_i = \left(f(y_i) - g(y_i)\right)\Delta y = \left(1 + \frac{3}{4}y_i - \frac{1}{4}y_i^2\right)\Delta y_i$$

$$A \approx \sum \Delta A_i = \cdots$$

$$\Big\Downarrow \text{ limit}$$

$$A = \int_{-1}^{4}\left(1 + \frac{3}{4}y - \frac{1}{4}y^2\right)dy = \frac{1}{4}\int_{-1}^{4}\left(4 + 3y - y^2\right)dy$$

$$= \frac{1}{4}\left[4y + \frac{3}{2}y^2 - \frac{1}{3}y^3\right]_{-1}^{4} = \frac{1}{24}\left[24y + 9y^2 - 2y^3\right]_{-1}^{4}$$

$$= \frac{1}{24}\left\{\left[96 + 144 - 128\right] - \left[-24 + 9 + 2\right]\right\} = \frac{1}{24}\left(112 - (-13)\right)$$

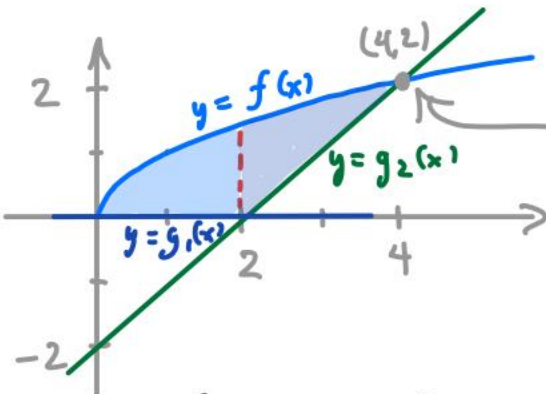
$$= \frac{125}{24} \quad (\approx 5.21),$$

which was substantially easier. //

In some cases, there really is no way around breaking the integral in two; but there is usually an easier way to do it (horizontal vs. vertical).

Ex/ Find the area of the region bounded by

$y = \sqrt{x}$, $y = x - 2$, and the x-axis.



$\sqrt{x} = x - 2$

$x = x^2 - 4x + 4$

$0 = x^2 - 5x + 4 = (x-4)(x-1)$

$x = 4, \cancel{1}$

Split it up!

$$A = \int_0^2 \left(f(x) - g_1(x)\right) dx + \int_2^4 \left(f(x) - g_2(x)\right) dx$$

$$= \int_0^2 x^{1/2}\, dx + \int_2^4 \left(x^{1/2} - x + 2\right) dx$$

$$= \left[\frac{x^{3/2}}{3/2}\right]_0^2 + \left[\frac{x^{3/2}}{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$

$$= \cancel{\frac{2}{3} \cdot 2^{3/2}} + \left(\frac{2}{3} 4^{3/2} - \frac{1}{2} 4^2 + 2 \cdot 4\right) - \left(\cancel{\frac{2}{3} 2^{3/2}} - \frac{1}{2} 2^2 + 2 \cdot 2\right)$$

$$= \frac{16}{3} - \cancel{8} + \cancel{8} - (-2 + 4) \quad = \frac{16}{3} - 2$$

$$= \frac{10}{3}.$$