Recall from last lecture that for a function \( f \) with domain \( D \) containing a punctured neighborhood of \( p \), and \( L \in \mathbb{R} \),
\[
\lim_{x \to p} f(x) = L
\]

means that

for each given \( \epsilon > 0 \), there exists some \( \delta > 0 \) such that
\[
0 < |x - p| < \delta \Rightarrow |f(x) - L| < \epsilon.
\]

\((x \in (p-\delta, p+\delta) \setminus \{p\}) \quad (f(x) \in (L-\epsilon, L+\epsilon))\)

Basic examples:
- \( f(x) = C \) (constant) — any \( \delta \) works
- \( f(x) = x \) — \( \delta = \epsilon \) works

Basic non-examples: did some of these last time. For instance,

Suppose \( \lim_{x \to 0} \frac{1}{x^2} = L \), for some real \( \neq L \). Then given (say) \( \epsilon = 1 \),
there exists \( \delta \) such that \( x \in (0, \delta) \Rightarrow \frac{1}{x^2} \in (L-1, L+1) \Rightarrow \frac{1}{x^2} < L+1 \)

\( \Rightarrow L+1 \in \mathbb{R}^+ \) and \( x > \frac{1}{\sqrt{L+1}} \), which is a contradiction b/c we took any \( x \in (0, \delta) \)!

How do we get more examples? By the

**LIMIT LAWS**

Suppose \( \lim_{x \to p} f(x) = L \), \( \lim_{x \to p} g(x) = b \) exist.

\( \begin{align*}
\text{A} \quad & \lim_{x \to p} (f(x) + g(x)) = \lim_{x \to p} f(x) + \lim_{x \to p} g(x) \\
\text{B-\(C) \quad & \lim_{x \to p} \frac{f(x)}{g(x)} = \lim_{x \to p} \frac{f(x)}{g(x)} \lim_{x \to p} \frac{g(x)}{g(x)} \quad \text{if } \lim_{x \to p} g(x) \neq 0 \}, \lim_{x \to p} f(x) / \lim_{x \to p} g(x) \text{ provided }\}
\text{D} \quad & \text{If } f \leq H \leq g \text{ for } x \in N^*(p), \text{ and } L_f = L = L_g, \text{ then } \lim_{x \to p} H(x) = L.
\end{align*} \)
Proof of (A): Let \( \varepsilon > 0 \) be given. Then exist

- \( \delta_f \) s.t. \( 0 < |x-p| < \delta_f \Rightarrow |f(x) - L_f| < \frac{\varepsilon}{2} \)
- \( \delta_g \) s.t. \( 0 < |x-p| < \delta_g \Rightarrow |g(x) - L_g| < \frac{\varepsilon}{2} \).

So if I take \( \delta = \min \{ \delta_f, \delta_g \} \), both of these hold, and

\[
|f(x) + g(x) - (L_f + L_g)| = |(f(x) - L_f) + (g(x) - L_g)| \\
\leq |f(x) - L_f| + |g(x) - L_g| \\
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

(\( \square \))

Proof of (B): Let \( \varepsilon > 0 \) be given, and put \( C_\varepsilon = \min \{ \frac{\varepsilon}{1 + |L_f| + |L_g|}, 1 \} \). As above, there exists a "common" \( \delta \) s.t. \( 0 < |x-p| < \delta \Rightarrow 
\mathrm{ |f(x) - L_f|, |g(x) - L_g| < C_\varepsilon .}
\)

So then

\[
|f(x)g(x) - L_f L_g| = |(f-L_f)g + (g-L_g)L_f| \\
= |(f-L_f)(g-L_g) + L_g(f-L_f) + L_f(g-L_g)| \\
\leq |f-L_f| |g-L_g| + |L_g||f-L_f| + |L_f||g-L_g| \\
< C_\varepsilon^2 + |L_g| C_\varepsilon + |L_f| C_\varepsilon \\
= C_\varepsilon (C_\varepsilon + |L_g| + |L_f|) \\
\leq C_\varepsilon (1 + |L_g| + |L_f|) \\
\leq \frac{\varepsilon}{1 + |L_g| + |L_f|} (1 + |L_g| + |L_f|) = \varepsilon.
\]

(\( \square \))

Application: Define \( f(x) \) to be continuous at a point \( p \in \mathbb{R} \)

- (i) \( \lim_{x \to p} f(x) \) exists,
- (ii) \( f(p) \) exists \((p \in \mathbb{R})\), and
- (iii) \( \lim_{x \to p} f(x) = f(p) \).

\( f(x) \) is "continuous" if it is continuous at all points in its domain \( \mathbb{D} \).

A-B) say that sums \& products of continuous functions are continuous.

Since \( f(x), C \) \& \( f(x) \cdot x \) are continuous, from this we get that all polynomial functions are continuous!
Proof of (C): If we show \( \lim_{x \to p} \frac{f(x)}{g(x)} = L \), we can just apply (B).

Given \( \varepsilon > 0 \), let \( \varepsilon = \min \left\{ \varepsilon \left( \frac{1}{2} \right), \frac{1}{2} \right\} \) and pick \( \delta > 0 \) s.t.
\[
0 < |x-p| < \delta \implies \left| \frac{f(x)}{g(x)} - L \right| < \varepsilon.
\]

Then
\[
\left| \frac{f(x)}{g(x)} - L \right| < \frac{1}{2} \implies \left| \frac{f(x)}{g(x)} - L \right| \leq \frac{1}{2},
\]
and
\[
\left| \frac{1}{g(x)} - \frac{1}{L} \right| = \frac{|L - g|}{|g||L|} < \frac{\varepsilon}{L} \leq \frac{\varepsilon}{\frac{1}{2}} = \varepsilon.
\]

Application of (C): Quotients \( \frac{f(x)}{g(x)} \) of continuous functions are continuous on the complement of the set where \( g(x) = 0 \). Since polynomials are continuous, so therefore are rational functions \( \frac{P(x)}{Q(x)} \) where \( Q(x) \neq 0 \).

Proof of (D): Given \( \varepsilon > 0 \), \( \exists \ \delta > 0 \) s.t. \( 0 < |x-p| < \delta \implies \left| f(x) - L \right| < \frac{\varepsilon}{3} \). Since \( f \leq H \leq g \implies 0 \leq g-|f| \leq g-f \), this yields
\[
\left| H(x) - L \right| = \left| (H-g)+(g-L) \right| \leq \left| H-g \right| + \left| g-L \right| \leq \left| g-f \right| + \left| g-L \right| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\]

Application of (D): \( \cos x < x < \frac{1}{\cos x} \) for \( x \in \mathbb{R} \setminus (0; \pi/2) \).

We'll show in a moment that \( \cos(x) \) is continuous; so \( \lim_{x \to 0} \cos(x) = 1 \).

From (C) it follows that \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \). So by (D),
\[
\lim_{x \to 0} \frac{\sin x}{x} = 1.
\]

The following result gives (together with \( C \)) the continuity of all fractional powers and trigonometric functions where they are defined, since they all arise as iterated compositions of bounded functions.
Theorem: If \( f(x) \) is integrable on \([a,b]\), \( F(x) := \int_a^x f(t) \, dt \) is continuous on \([a,b]\).

Proof: Since \( f \) is integrable, it is bounded, so \( \exists M \in \mathbb{R}^+ \) with \( M \geq |f(x)| \) on \([a,b]\). Thinking of \(-M \leq f \leq M\) as lower and upper step functions, we get

\[
M |x-p| \geq \left| \int_p^x f(t) \, dt \right| = |F(x) - F(p)|.
\]

Given \( \epsilon > 0 \), we take \( \delta := \frac{\epsilon}{M} \); this yields

\[
0 < |x-p| < \delta \implies |F(x) - F(p)| \leq M |x-p| < M \delta = M \frac{\epsilon}{M} = \epsilon.
\]

\( \square \)

Remark: We can also talk about left or right limits and left or right continuity: simply replace everywhere

- for "left": \( \lim_{x \to p^-} F(x) \), \( 0 < |x-p| < \delta \) by \( x \in (p-\delta, p) \)
- for "right": \( \lim_{x \to p^+} F(x) \), \( 0 < |x-p| < \delta \) by \( x \in (p, p+\delta) \).

For instance, the greatest integer function

\[
y = [x]
\]

has \( \lim_{x \to 1^+} [x] = 1 \), \( \lim_{x \to 1^-} [x] = 0 \).

Since \([1] = 1\), \([x]\) is right-continuous but not left-continuous at 1. In general, "continuity" is the same as "left-continuity" + "right-continuity"; and existence of \( \lim_{p \uparrow} f(x) \) is the same as existence and equality of the left & right limits.

Reason for this is simply that \( \mathbb{W}^* (p; \delta) = (p-\delta, p) \cup (p, p+\delta) \).

\( \uparrow \)

used to define limit

\( \uparrow \)

used to define left limit

\( \uparrow \)

used to define right limit