Lecture 12: Limits & Continuity

Recall from last lecture that for a function \( f \) with domain \( D \) containing a punctured neighborhood of \( p \), and \( L \in \mathbb{R} \),

\[
\lim_{x \to p} f(x) = L
\]

means that

for each given \( \epsilon > 0 \), there exists some \( \delta > 0 \) such that

\[
0 < |x - p| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.
\]

\((x \in (p-\delta, p+\delta) \setminus \{p\}) \quad (f(x) \in (L-\epsilon, L+\epsilon))\)

Basic examples: \( f(x) = C \) (constant) — any \( \delta \) works

\( f(x) = x \) — \( \delta = \epsilon \) works

Basic non-examples: did some of these last time. For instance,

Suppose \( \lim_{x \to 0} \frac{1}{x^2} = L \), for some real \( \neq L \). Then given \( \epsilon = 1 \),

there exists \( \delta \) s.t. \( x \in (0, \delta) \Rightarrow \frac{1}{x^2} \in (L-1, L+1) \Rightarrow \frac{1}{x^2} < L+1 \)

\( \Rightarrow L+1 \in \mathbb{R}^+ \) and \( x > \frac{1}{\sqrt{L+1}} \), which is a contradiction b/c we took any \( x \in (0, \delta) \)!

How do we get more examples? By the

**LIMIT LAWS**

Suppose \( \lim_{x \to p} f(x) = L \), \( \lim_{x \to p} g(x) = L \) exist.

(A) \( \lim_{x \to p} (f(x) + g(x)) = \lim_{x \to p} f(x) + \lim_{x \to p} g(x) \)

(B) \( \lim_{x \to p} c f(x) = c \lim_{x \to p} f(x) \), \( \lim_{x \to p} \frac{f(x)}{g(x)} = \frac{\lim_{x \to p} f(x)}{\lim_{x \to p} g(x)} \) provided the denominator is non-zero.

(C) \( \lim_{x \to p} \min(g(x)) = \lim_{x \to p} \sup(g(x)) = \min(L, L) \), \( \sup(L, L) \), then \( \lim_{x \to p} H(x) = L \).
Proof of (A): Let \( \varepsilon > 0 \) be given. Then exist
\[
\delta_f \text{ s.t. } 0 < |x-p| < \delta_f \Rightarrow |f(x) - L_f| < \frac{\varepsilon}{2},
\]
\[
\delta_g \text{ s.t. } 0 < |x-p| < \delta_g \Rightarrow |g(x) - L_g| < \frac{\varepsilon}{2}.
\]
So if I take \( \delta := \min \{ \delta_f, \delta_g \} \), both of these hold, and
\[
|(f(x) + g(x)) - (L_f + L_g)| = |(f(x) - L_f) + (g(x) - L_g)|
\]
\[
\leq |f(x) - L_f| + |g(x) - L_g| \quad \text{(triangle inequality)}
\]
\[
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Proof of (B): Let \( \varepsilon > 0 \) be given, and put \( C_\varepsilon := \min \{ \frac{\varepsilon}{1+|L_f|+|L_g|}, 1 \} \).
As above, there exists a "common" \( \delta \) s.t. \( 0 < |x-p| < \delta \Rightarrow |f(x) - L_f|, |g(x) - L_g| < C_\varepsilon \).
So then
\[
|f(x)g(x) - L_fL_g| = |(f-L_f)g + (g-L_g)L_f|
\]
\[
= |(f-L_f)(g-L_g) + L_g(f-L_f) + L_f(g-L_g)|
\]
\[
\leq |f-L_f||g-L_g| + |L_g||f-L_f| + |L_f||g-L_g|
\]
\[
< C_\varepsilon^2 + |L_g|C_\varepsilon + |L_f|C_\varepsilon
\]
\[
= C_\varepsilon(C_\varepsilon + |L_g| + |L_f|)
\]
\[
\leq C_\varepsilon(C_\varepsilon + |L_g| + |L_f|)
\]
\[
\leq \frac{\varepsilon}{1+|L_g|+|L_f|} (1+|L_g|+|L_f|) = \varepsilon.
\]

Application: Define \( f(x) \) to be continuous at a point \( p \in \mathbb{R} \)
if (i) \( \lim_{x \to p} f(x) \) exists, (ii) \( f(p) \) exists (\( p \in D \)), and (iii) \( \lim_{x \to p} f(x) = f(p) \).
\( f(x) \) is "continuous" if it is continuous at all points in its domain \( D \).

A-B say that sums and products of continuous functions are continuous.
Since \( f(x)c \) and \( f(x)x \) are continuous, from this we get that
all polynomial functions are continuous!
Proof of (C): If we show \( \lim_{x \to p} g(x) = L \), we can just apply (B). Given \( \varepsilon > 0 \), let \( C = \min \{ \frac{\varepsilon L}{2}, \frac{\varepsilon L}{2} \} \) and pick \( \delta > 0 \) s.t.

\[
0 < |x-p| < \delta \implies |g(x) - L| < C.
\]

Then

\[
|g(x) - L| < \frac{L \varepsilon}{2} \implies |g(x)| > \frac{L \varepsilon}{2},
\]

and

\[
\left| \frac{1}{g(x)} - \frac{1}{L} \right| = \frac{|L - g|}{|g||L|} < \frac{C}{L||g||} = \frac{\varepsilon L^2}{2} \leq \frac{\varepsilon L \cdot \frac{L \varepsilon}{2}}{L \cdot \frac{L \varepsilon}{2}} = \varepsilon.
\]

Application of (C): Quadrants \( f(x), g(x) \) of continuous functions are continuous on the complement of the set where \( g(x) = 0 \). Since polynomials are continuous, so therefore are rational functions \( \frac{f(x)}{g(x)} \) where \( g(x) \neq 0 \).

Proof of (D): Given \( \varepsilon > 0 \), \( \exists \delta > 0 \) s.t. \( 0 < |x-p| < \delta \implies |f(x) - L| < \varepsilon/3 \). Since \( f \leq h \leq g \implies 0 \leq g-h \leq g-f \), this yields

\[
|H(x) - L| = |(H-g) + (g-L)| \leq |H-g| + |g-L|
\]

\[
\leq |g-f| + |g-L| \leq |g-L| + |L-f| + |f-L|
\]

\[
< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \quad \square
\]

Application of (D): \( \cos x < \frac{\sin x}{x} < \frac{1}{\cos x} \) for \( x \in N^x(0; \pi/2) \).

We'll show in a moment that \( \cos(x) \) is continuous; so \( \lim_{x \to 0} \cos(x) = 1 \). From (C) it follows that \( \lim_{x \to 0} \frac{1}{\cos(x)} = 1 \). So by (D), \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \).

The following result gives (together with (C)) the continuity of all fractional powers and trigonometric functions where they are defined, since they all arise as integrals of bounded functions.
Theorem: If \( f(x) \) is integrable on \([a,b]\), \( F(x) = \int_a^x f(t) \, dt \) is continuous on \([a,b]\).

Proof: Since \( f \) is integrable, it is bounded, so \( \exists M \in \mathbb{R}^+ \) with \( M \geq |f(x)| \) on \([a,b]\). Thinking of \(-M \leq f \leq M\) as lower \& upper step functions, we get
\[
M |x-p| \geq \left| \int_p^x f(t) \, dt \right| = \left| F(x) - F(p) \right|.
\]

Given \( \varepsilon > 0 \), we take \( \delta := \frac{\varepsilon}{M} \) \( \Rightarrow \) this yields
\[
0 < |x-p| < \delta \Rightarrow \left| F(x) - F(p) \right| \leq M |x-p| < M \delta = M \frac{\varepsilon}{M} = \varepsilon.
\]

Remark: We can also talk about left or right limits \&
left or right continuity: simply replace everywhere
- for "left": \( \lim_{x \to p^-} \), \( 0 < |x-p| < \delta \) by \( x \in (p-\delta, p) \)
- for "right": \( \lim_{x \to p^+} \), \( 0 < |x-p| < \delta \) by \( x \in (p, p+\delta) \).

For instance, the greatest integer function
\[
\begin{array}{c}
\text{has } \lim_{x \to 1^+} [x] = 1, \lim_{x \to 1^-} [x] = 0.
\end{array}
\]

Since \([1]=1\), \([x]\) is right-continuous but not left-continuous at 1. In general, "continuity" is the same as "left-continuity" \+ "right-continuity"; and existence of \( \lim_{x \to p} f(x) \) is the same as existence and equality of the left \& right limits.

Reason for this is simply that \( W^+(p; \delta) = (p-\delta, p) \cup (p, p+\delta) \).

\\[ \uparrow \text{used to define limit} \quad \uparrow \text{used to define left limit} \quad \uparrow \text{used to define right limit} \]