Lecture 13: More on continuous functions

Ex: How do we extend \( F(x) := \frac{\sin(2x)}{x} \) (on \( \mathbb{R} \setminus \{0\} \)) to a continuous function on \( \mathbb{R} \)?

Ideally, we would write \( \frac{\sin(2x)}{x} = \frac{\sin(2x)}{2x} \cdot 2 \) and then argue that \( \lim_{x \to 0} \frac{\sin(2x)}{2x} \) is the “same” as \( \lim_{y \to 0} \frac{\sin(y)}{y} \). While this is intuitively clear, and you could write out a limit argument, it seems better to prove a more general result which Apostol seems to overlook.

Definition: Let \( f \) be a function with domain \( D \), and \( g \) be a function with domain \( E \). The composition \( f \circ g \) is defined by \( (f \circ g)(x) := f(g(x)) \), and has domain given by \( \text{Dom}(f \circ g) = \{ x \in E \mid g(x) \in D \} \).

Theorem: (Limit law for compositions)

(i) If \( \lim_{x \to p} g(x) \) is defined (or and not have \( p \in E \)) and \( f \) is continuous at \( \lim_{x \to p} g(x) \), then \( \lim_{x \to p} (f \circ g)(x) = f(\lim_{x \to p} g(x)) \).

(ii) If \( g \) is continuous at \( p \), and \( \lim_{y \to g(p)} f(y) \) exists, then \( \lim_{x \to p} (f \circ g)(x) = \lim_{y \to g(p)} f(y) \) provided that \( g(x) \neq g(p) \) for \( x \in N^*(p) \) (i.e. \( x \) near but not equal to \( p \)).

Proof: (i) Write \( h := \lim_{x \to p} g(x) \). By hypothesis, \( \lim_{y \to h} f(y) = f(h) \).

So given \( \varepsilon > 0 \), there exists \( \delta > 0 \) s.t.

\[
|y - h| < \delta \implies |f(y) - f(h)| < \varepsilon.
\]

Moreover, thinking of \( \varepsilon, \delta \) as “the \( \varepsilon \)” for \( g(x) \), there exists \( \delta > 0 \) s.t.

\[
0 < |x - p| < \delta \implies |g(x) - h| < \varepsilon.
\]
But then taking \( y = g(x) \) in \((**)*\), we get \( |f(g(x)) - f(y)| < \varepsilon \) as desired. So \( \lim_{x \to p} f(g(x)) = f(g(p)) \).

\((**i)\) Write \( L := \lim_{y \to g(p)} f(y) \), and let \( \varepsilon > 0 \) be given by defn. of limit \( \exists \delta > 0 \) s.t.

\[ 0 < |y - g(p)| < \delta \implies |f(y) - L| < \varepsilon. \]

By continuity of \( f \) together with the additional hypothesis, \( \exists \delta > 0 \) s.t.

\[ 0 < |x - p| < \delta \implies 0 < |g(x) - g(p)| < \varepsilon. \]

Taking \( y = g(x) \) in \((**i)\), we get \( |f(g(x)) - L| < \varepsilon \), done. \( \Box \)

Now write \( F(x) = f(g(x)) \), where \( g(x) = 2x \) and \( f(y) = 2 \frac{\sin(y)}{y} \).

Applying \((**i)\), \( \lim_{x \to 0} F(x) = \lim_{y \to g(0)} f(y) = \lim_{y \to 0} 2 \frac{\sin(y)}{y} = 2 \cdot 1 = 2. \)

**Corollary (of the Theorem)**: If \( g \) is continuous at \( p \) and \( f \) is continuous at \( g(p) \), then \( f \circ g \) is continuous at \( p \).

(This is just \((**i)\) in the special case where \( g \) is continuous at \( p \).)

**Ex/** Let \( f(x) = \sin(x) \), \( g(x) = \sqrt{x} \) with domain \( \mathbb{R} \geq 0 \).

\( (f \circ g)(x) = \sin(\sqrt{x}) \) is continuous w/ domain \( \mathbb{R} \geq 0 \).

\( (g \circ f)(x) = \sqrt{\sin(x)} \) is continuous w/ domain \( U \left[ 2k\pi, (2k+1)\pi \right], \) \( k \in \mathbb{Z} \).

You should convince yourself they aren't the same!

Composition is not a commutative operation: \( f \circ g \neq g \circ f \) in most cases. But it is associative in the sense that \( (f \circ g) \circ h = f \circ (g \circ h) \). \( \Box \)
Intermediate Value Theorem

Suppose you’re traveling on the highway, starting before mile marker 100 at noon and ending up well past mile marker 200 at 2 pm. There is a cop car starting at mile marker 100 at noon and driving a constant speed until 2 pm. The fact that you are definitely going to get radared is a consequence of the following (opposed to the difference g−h):

Bolzano’s Theorem: If f is continuous on [a, b], and f(a) and f(b) have opposite signs, then ∃c ∈ (a, b) s.t. f(c) = 0.

Lemma: If f is continuous at c and f(c) ≠ 0, then ∃δ > 0 s.t. |x − c| < δ ⇒ f(x) has the same sign as f(c).

Proof: We may assume f(c) > 0 (why?). Pick ε = f(c)/2, then (by the continuity assumption) ∃δ > 0 s.t. |x − c| < δ ⇒ |f(x) − f(c)| < ε.

But then f(x) ∈ (f(c) − ε, f(c) + ε) = (f(c) − f(c) + f(c)/2) > f(x) − f(c) − c = f(c)/2 > 0.

Proof of Bolzano: Assume f(a) < 0, f(b) > 0. Let S = {x ∈ [a, b] | f(x) ≤ 0}. It is nonempty and bounded above, so c = sup S exists.

• Suppose f(c) > 0; then by the lemma, ∃δ > 0 s.t. f > 0 on (c − δ, c + δ) ∩ [a, b].
• Suppose f(c) < 0. Then f < 0 on some (c − δ, c + δ) (by the lemma), but then (easy) c + δ/2 ∈ S contradicting fact that c is an UB for S.
• Hence f(c) = 0.
Intermediate Value Theorem: If $f$ is continuous on $[a,b]$, then $f$ assumes every value $y_0$ between $f(a)$ and $f(b)$ somewhere in $(a,b)$.

Proof: Apply Bolzano to $f - y_0$.

Remark: Continuity on $[a,b]$ means in particular that $f$ is right-continuous at $a$ and left-continuous at $b$. Without continuity, Bolzano & IVT fail! For example, let $f(x) := \tan(x)$: we have $f(\pi/4) = 1$ and $f(3\pi/4) = -1$, but nowhere in $(\pi/4, 3\pi/4)$ does $f(x) = 0$.

An application: Inverse functions

Proposition: Let $f$ be a continuous, strictly monotone function on $[a,b]$. Then there is a continuous, strictly monotone function $g$ on $[f(a), f(b)]$ (or $[p(e), f(a)]$) such that $(g \circ f)(x) = x$.

Proof: Given any $y_0 \in I$, by the IVT there is some $x_0 \in [a,b]$ at which $f(x_0) = y_0$. By strict monotonicity, this $x_0$ is unique. So we can define $g(y_0) := x_0$. The argument that $g$ is strictly monotone is routine. For continuity at $y_0$, let $\varepsilon > 0$ be given (and assume $f$ increasing). Pick $\delta := \min \{y_0 - f(x_0 - \varepsilon), f(x_0 + \varepsilon) - y_0\}$. Then $y \in (y_0 - \delta, y_0 + \delta) \Rightarrow y \in (f(x_0 - \varepsilon), f(x_0 + \varepsilon)) \Rightarrow g(y) \in (x_0 - \varepsilon, x_0 + \varepsilon) = (g(y_0) - \varepsilon, g(y_0) + \varepsilon)$. \qed
This $g$, called the inverse of $f$, has graph $\Gamma_g$ given by the reflection of $\Gamma_f$ through the diagonal $y = x$ line.

Why is this? Notice that $y = g(x)$ is the same as $(x, y)$ applied to both sides of flip around $y = x$. That is, $\Gamma_g = \{ (x, y) \mid (y, x) \in \Gamma_f \}$. The figure explains why swapping $x$ and $y$ gives the reflection.

Remark: We will sometimes write $g = f^{-1}$. (But this can also be a confusing notation, since $f^{-1}$ can also mean $\frac{1}{f}$.) For instance, if $f(x) = x^2$, then $f^{-1}(x) = \sqrt{x}$.

Example: Find $f^{-1}$ if $f(x) = 2x + 5$. Then $y = f(x)$, $x = f^{-1}(y)$. So I need to solve $y = 2x + 5$ for $x$: $y - 5 = 2x$ 

\[ \frac{1}{2}(y - 5) = x \]

\[ \Rightarrow f^{-1}(y) = \frac{1}{2}(y - 5). \]