Lecture 15: (Re) introduction to derivatives

In the last lecture we showed that continuous functions on a closed interval \((a,b)\) are integrable and (b) actually attain their extreme and average values at some point on the interval. Today's theoretical content is mild by comparison, though we will see that (c) differentiable functions are continuous; by (c1) this means they are also integrable, which sets us up nicely for the Fundamental Theorem of Calculus next week.

**Definitions:** (1) If \(Dom(f)\) contains a neighborhood of \(p\), and \(f'(p) = \lim_{h \to 0} \frac{f(p+h) - f(p)}{h}\) exists, then \(f\) is said to be differentiable at \(p\) and \(f'(p)\) is the derivative at \(p\); it is said to be differentiable on \((a,b)\) if it is differentiable at every \(p \in (a,b)\), and then \(f'(a)\) only on the status of a function. (We can also talk about right & left differentiability at the endpoints.)

(2) The tangent line to the graph \(f\) at \((p, f(p))\) is defined to be the line with equation

\[
y - f(p) = f'(p) \cdot (x - p).
\]

(This is motivated by the picture at the end of the notes for Lecture 14.)
Theorem 1: Differentiability (at $p$) implies continuity (at $p$).

Proof: If $f$ is differentiable at $p$, then $\lim_{x \to p} \frac{f(x) - f(p)}{x - p} = f'(p)$ by the limit law for compositions (i.e. substituting $h = x - p$). Choose $\delta_0 > 0$ s.t. $0 < |x - p| < \delta_0 \Rightarrow \frac{f(x) - f(p)}{x - p} \in (f'(p) - 1, f'(p) + 1)

\Rightarrow |\frac{f(x) - f(p)}{x - p}| < |f(p)| + 1 =: M \Rightarrow |f(x) - f(p)| < M|x - p|.

Now let $\varepsilon > 0$ be given. Choosing $\delta := \min \{\delta_0, \frac{\varepsilon}{M}\}$

$\Rightarrow |f(x) - f(p)| < M|x - p| < M \frac{\varepsilon}{M} = \varepsilon$. \qed

Notation for derivatives: take $y = f(x)$.

- Newton: $\dot{y}$, $\ddot{y}$, ... (used in physics, e.g. classical mechanics)
- Leibniz: $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, ...
- Arbogast: $Df$, $D^2f$, ... (used in differential equations)
- Lagrange: $f'(x)$, $f''(x)$, ...

Differentiation rules:

(i) $(f \pm g)' = f' \pm g'$

(ii) $(fg)' = f'g + g'f$

(iii) $\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$ (where $g \neq 0$)

Proofs: (i) is immediate from limit law for "\pm":

$$\lim_{h \to 0} \frac{f(x + h) \pm g(x + h) - (f(x) \pm g(x))}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x) \pm g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \pm \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

(iii) follows from (ii) once you know $\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$, since then

$$(f \cdot \frac{1}{g})' = f' \cdot \frac{1}{g} + f \cdot \left(\frac{1}{g}\right)' = \frac{f'g}{g^2} - \frac{fg'}{g^2} = \frac{f'g - fg'}{g^2}.$$
\[
\left( \frac{1}{g} \right)' = \lim_{h \to 0} \frac{1}{g(x+h)} - \frac{1}{g(x)} = \lim_{h \to 0} \frac{g(x) - g(x+h)}{h g(x+h) g(h)} \\
= -\lim_{h \to 0} \frac{g(x+h)-g(x)}{h} \cdot \lim_{h \to 0} \frac{1}{g(x+h) g(h)} = -\frac{g'(x)}{(g(x))^2}.
\]

(ii): \[
\lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\
= \lim_{h \to 0} \frac{g(x+h) \cdot \frac{f(x+h) - f(x)}{h}} + \lim_{h \to 0} f(x) \cdot \frac{g(x+h) - g(x)}{h} \\
= (\lim_{h \to 0} g(x+h)) \cdot (\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}) + f(x) \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = g(x) \cdot f'(x) + f(x)g'(x).
\]

\[\square\]

Ex 1/ \( f(x) = c \Rightarrow f' = 0 \): \[\lim_{h \to 0} \frac{c-c}{h} = \lim_{h \to 0} 0 = 0 \]

Ex 2/ \( f(x) = x \Rightarrow f' = 1 \): \[\lim_{h \to 0} \frac{x+h-x}{h} = \lim_{h \to 0} 1 = 1 \]

Ex 3/ Taking Ex 2 as the base case, we show \( A(n) : \frac{d}{dx} x^n = nx^{n-1} \) by induction: assuming \( A(k) \), \( \frac{d}{dx} x^{k+1} = \frac{d}{dx} (x^k \cdot x) = kx^{k-1} \cdot x + x^k \Rightarrow A(k+1) \).

Together with (i), this gives \( \frac{d}{dx} \sum_{k=0}^{n} a_k x^k = \sum_{k=1}^{n} k a_k x^{k-1} \) for polynomials; and \( A(iii) \) allows us to differentiate rotated fences.

Ex 4/ \( f(x) = x^{1/n} \). While \( u(h) := (x+h)^{1/n} \), so that (by continuity)

\[\lim_{h \to 0} u(h) = u(0) = x^{1/n} \text{ and } u(h)^n - u(0)^n = h. \]

So \( u'(x) = \lim_{h \to 0} \frac{(x+h)^{1/n} - x^{1/n}}{h} = \lim_{h \to 0} \frac{u(h)^n - u(0)^n}{h u(0)^{n-1}} \)

\[= \lim_{h \to 0} \frac{u(h)^{n-1} u(h)^{-1} + u(h)^{n-2} u(0) + \ldots + u(0)^{n-1}}{u(0)^{n-1}} \]

\[= \frac{1}{u(0)^{n-1}} = \frac{1}{n \cdot x^{1/n-1}}. \]

**Limit**
As in Ex 3, induction extends the power rule law to $x^{\frac{m}{n}}$ and hence to all rational powers.

Ex 5. $f(x) = \sin(x):$ Recall $\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right) \sin\left(\frac{a-b}{2}\right)$.

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{h \to 0} \frac{2\cos\left(x + \frac{h}{2}\right) \sin\left(\frac{h}{2}\right)}{h} = \lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \cdot \lim_{h \to 0} \cos\left(x + \frac{h}{2}\right) = 1 \cdot \cos(x) = \cos(x).$$

$f(x) = \cos(x):$ Use $\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right) \sin\left(\frac{a-b}{2}\right)$ in the same way. Note that the way these are done in other calculus books uses $
\lim_{h \to 0} \frac{1-\cos(h)}{h} = \lim_{h \to 0} \frac{\cos(0) - \cos(h)}{h} = \lim_{h \to 0} \frac{-2\sin\left(\frac{h}{2}\right) \sin\left(-\frac{h}{2}\right)}{h} = \lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \cdot \lim_{h \to 0} \cos\left(\frac{h}{2}\right) = 0 \cdot 1 = \sin(0) = 0$.

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{h \to 0} \frac{\sin(x) \cos(h) + \cos(x) \sin(h) - \sin(x)}{h} = \sin(x) \lim_{h \to 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h \to 0} \frac{\sin(h)}{h} = \cos(x).$$

Ex 6. $f(x) = \cot(x) = \frac{\cos(x)}{\sin(x)}$.

By (iii), $f'(x) = \frac{\sin(2x) \cos(x) - \cos(x) \sin(x)}{\sin^2(x)} = \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)} = \frac{-1}{\sin^2(x)} = -\csc^2(x)$.

Ex 7. Derivatives are rates of change. This is reflected by the fact that the rate of change of the area of a disk w.r.t. change in radius is...
\[
\frac{dA}{dr} = \frac{d}{dr} \pi r^2 = 2\pi r = \text{circumference},
\]
and the rate of change of the volume of a ball is
\[
\frac{dV}{dr} = \frac{d}{dr} \frac{4}{3} \pi r^3 = 4\pi r^2 = \text{surface area}.
\]

Ex 8/ Claim: \( y = -x \) is tangent to \( y = x^3 - 6x^2 + 8x \).

To find intersection points, write \(-x = x^3 - 6x^2 + 8x \)
\( \Rightarrow 0 = x^3 - 6x^2 + 9x = x(x-3)^2 \Rightarrow x = 0, 3 \)
\( \Rightarrow (x, y) = (0, 0) \) and \( (3, -3) \).

The line has slope \(-1\). The curve has slope \( f'(x) = 3x^2 - 12x + 8 \) which is 8 at \((0, 0)\)
and \(-1\) at \((3, -3)\). So indeed, the line is tangent to the curve at \((3, -3)\). In a picture: