Lecture 16: Mean and Extreme, part II

Locating extreme values of $f : D^{\dagger} \to \mathbb{R}$

Definition 1: f has a $\begin{cases} \text{absolute maximum} \\ \text{absolute minimum} \\ \text{relative/local maximum} \\ \text{relative/local minimum} \end{cases}$ at c if

$$\begin{cases} f(c) \geq f(x) \text{ for all } x \in D \\ \exists\, N(c) \text{ s.t. } f(c) \geq f(x) \text{ for all } x \in N(c) \cap D. \end{cases}$$

Definition 2: c is a critical point of f if (i) $f'(c)$ DNE, (ii) $f'(c) = 0$, or (iii) c is an endpoint of D (i.e., $N(c) \cap D = [c, c+\epsilon)$ or $(c-\epsilon, c]$).

Theorem 1: Local extrema can only occur at critical points.

WARNING: a critical point need not be a local extremum!

Proof: Suppose f has a local extremum at c, and that (i) & (iii) do not hold — i.e. $f'(c)$ exists and $\exists\, N(c) \subset D$. Our job is to show that (ii) holds. Define $Q : N(c) \to \mathbb{R}$ by $Q(x) := \begin{cases} f'(c), & x = c \\ \frac{f(x)-f(c)}{x-c}, & x \neq c \end{cases}$. Since f is differentiable at c, Q is continuous at c.

If $f'(c) \gtrless 0$, then there exists a sub-neighborhood $(c-\epsilon, c+\epsilon) \subset N(c)$ on which $Q \lessgtr 0$. (Use continuity!) Hence $f(x) - f(c)$ and $x - c$ have the same / opposite sign on $(c-\epsilon, c+\epsilon)$, so that $f(x) > f(c)$ for $x \in (c, c+\epsilon)$ / $(c-\epsilon, c)$ and $f(x) < f(c)$ for $x \in (c-\epsilon, c)$ / $(c, c+\epsilon)$. Contradiction. □

So to find extrema (global or local), the first step is to find all critical points. But we'll need some more tools to determine which critical points are actual extrema.

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† assumed to be a finite union of intervals (open or closed)

Note: of course, an absolute extremum is also a local extremum.

Mean-value theorems for derivatives, applications

Let $h(x)$ be continuous on $[a,b]$, differentiable on (a,b). (*)

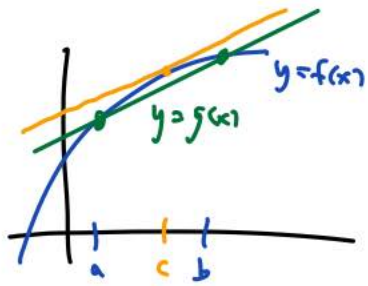
Lemma (Rolle's Theorem): If $h(a)=h(b)$, then $\exists\, c\in(a,b)$ s.t. $h'(c)=0$.

Proof: h continuous $\Rightarrow \exists\, d,e\in[a,b]$ s.t. $\begin{cases} M := h(e) = \sup\{h(x)\mid x\in[a,b]\} \\ m := h(d) = \inf\{h(x)\mid x\in[a,b]\}. \end{cases}$

Now assume there is no such c, i.e. $h'(x)\neq 0$ on $[a,b]$. Then the only critical pts. are $a\,\&\,b$, so by Theorem 1, $\{d,e\}\subseteq\{a,b\}$. Hence $\{m,M\}\subseteq\{h(a),h(b)\}=\{h(a)\}\Rightarrow m=M$

$\Rightarrow h$ constant $\Rightarrow h'$ identically zero, contradicting our assumption. □

Theorem 2 (MVT for derivatives): For any $f(x)$ satisfying (*), there exists a $c\in(a,b)$ s.t. $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.



Proof: Set $g(x) := f(a)+\dfrac{f(b)-f(a)}{b-a}(x-a)$, $h(x) := f(x)-g(x)$.

Then h satisfies (*) and $h(b)=h(a)$, so Rolle $\Rightarrow$

$\exists\, c\in(a,b)$ s.t. $0=h'(c)=f'(c)-\dfrac{f(b)-f(a)}{b-a}$. □

Corollary 1 (Cauchy MVT): Given F,G satisfying (*), $\exists\, c\in(a,b)$ s.t.

$$F'(c)\,(G(b)-G(a)) = G'(c)\,(F(b)-F(a)).$$ [Think: $\dfrac{F'(c)}{G'(c)}=\dfrac{\Delta F}{\Delta G}$.]

Proof: Take $h(x) = (G(b)-G(a))\,F(x) - (F(b)-F(a))\,G(x)$ and apply Rolle. □

Corollary 2: If h' is $\begin{cases} > \\ < \\ \geq \\ \leq \\ = \end{cases}$ 0 on (a,b), then h is $\begin{cases} \text{strictly increasing} \\ \text{strictly decreasing} \\ \text{increasing} \\ \text{decreasing} \\ \text{constant} \end{cases}$ on $[a,b]$.

Proof: $y>x \xrightarrow[\text{on } [x,y]]{\text{Thm. 2}} \exists\, c\in(x,y)$ s.t. $h'(c)=\dfrac{h(y)-h(x)}{y-x} \Rightarrow h(y)\begin{cases} > \\ < \\ \geq \\ \leq \\ = \end{cases} h(x)$. □

Corollary 3: If $h'\begin{cases} > \\ < \end{cases} 0$ on (a,c), $\begin{cases} < \\ > \end{cases} 0$ on (c,b), then h has a $\begin{cases} \text{maximum} \\ \text{minimum} \end{cases}$ at c. (Why?)

Ex 1/ Under the condition $x+y=S$ (on $x,y \in \mathbb{R}_{\geq 0}$), maximize xy:

Consider $f(x) = x(S-x)$ on $[0,S]$. critical pts. are $0, S$, and $\frac{S}{2}$ which solves $0 = f'(x) = S-2x$. But $f(0)=f(S)=0$, so $f(\frac{S}{2}) = \frac{S^2}{4}$ is the (global) maximum. //

Ex 2/ Under the condition $xy = P$ (on $x,y \in \mathbb{R}^+$), minimize $x+y$:

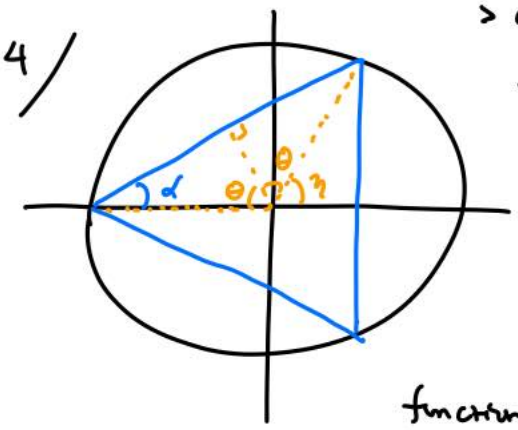
Consider $f(x) = x + \frac{P}{x}$ on $\mathbb{R}^+$, which has no "endpoints". In this kind of situation there need not be extrema! But solving $0 = f'(x) = 1 - \frac{P}{x^2}$ $\rightsquigarrow$ $x = \sqrt{P}$, we find that $f' > 0$ for $x > \sqrt{P}$ and $f' < 0$ for $x < \sqrt{P}$. By (essentially) Corollary 3, f has a (global) minimum at $\sqrt{P}$, namely $f(\sqrt{P}) = 2\sqrt{P}$. //

Ex 3/ To find the point on $x^2 = 4y$ closest to $(0,b)$, $b \in \mathbb{R}_{\geq 0}$, we minimize the square of the distance, $f(y) = [\text{dist}\{(0,b),(2\sqrt{y},y)\}]^2 = (2\sqrt{y})^2 + (y-b)^2$ $= y^2 + (4-2b)y + b^2$, on $\mathbb{R}_{\geq 0}$. Critical points are $\underline{0}$ (an endpt.) and $(0 = f'(y) = 2y + (4-2b) \Rightarrow y = \underline{b-2}$, if $b > 2$.

Case 1: $b \leq 2 \Rightarrow f' > 0$ for $y > 0 \Rightarrow$ (global) min at $0 \rightarrow (0,0)$ = closest pt.

Case 2: $b > 2 \Rightarrow f' \begin{matrix} < 0 \\ > 0 \end{matrix}$ for $y \begin{matrix} < b-2 \\ > b-2 \end{matrix} \Rightarrow$ (global) min at $b-2 \Rightarrow (2\sqrt{b-2}, b-2)$ = closest pt. //

Ex 4/



To maximize & minimize the perimeter of the isoceles triangle inscribed in the unit circle, with $\alpha \in [0, \pi/4]$, first note that $\Theta = \frac{\pi}{2} - \alpha$ and $\eta = 2\alpha$, so that the side lengths are $2\sin(2\alpha)$, $2\cos(\alpha)$, $2\cos(\alpha)$. So the perimeter function is $f(\alpha) = 2\sin(2\alpha) + 4\cos(\alpha)$ and

$0 = f'(\alpha) = 4\cos(2\alpha) - 4\sin(\alpha) \Rightarrow \sin(\alpha) = \cos(2\alpha) = 1 - 2\sin^2(\alpha)$

$\Rightarrow 2(\sin(\alpha))^2 + \sin(\alpha) - 1 = 0$

$\Rightarrow \sin(\alpha) = 1/2$ or $\underline{-1}$ ← not poss. for $\alpha \in [0, \pi/4)$

$\Rightarrow \alpha = \pi/6$.

Critical points are therefore $0, \pi/6, \pi/4$ at which $f(0) = 4$, $f(\pi/6) = 3\sqrt{3} \sim 5.2$, $f(\pi/4) = 2+2\sqrt{2} \sim 4.8$.

So max is $3\sqrt{3}$ & min is 4. //

In the last example, we can also consider $f''(x) = -8\sin(2x) - 4\cos(x)$, which has $f''(\pi/6) = -6\sqrt{3} < 0 \overset{\text{Cor. 2}}{\Rightarrow} f'$ decreasing at $\pi/6$ ($f' = 0$ at $\pi/6$) $\Rightarrow$ $f' > 0$ on $(\pi/6 - \varepsilon, \pi/6)$ and $f' < 0$ on $(\pi/6, \pi/6 + \varepsilon) \Rightarrow$ local max at $\pi/6$.

More generally this sort of reasoning (Cor. 3, assuming f'' exists) leads to

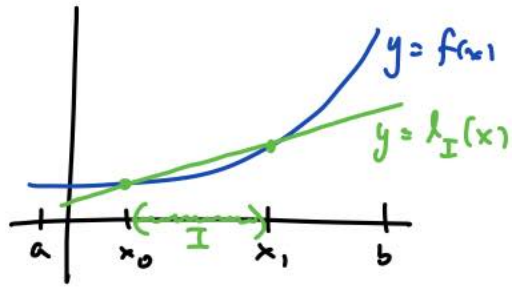
Corollary 4: If $f'(c) = 0$, $f''(c) \begin{cases} > \\ < \end{cases} 0$ in $N(c)$, then f has a local $\begin{cases} \min \\ \max \end{cases}$ at c.

Convexity and curve sketching

(also called "concave up" in Calculus textbooks)

Definition 3: f is convex on $[a,b]$ if, for every $x_0 < x_1$ in $[a,b]$, we have $f(x) \leq \ell_I(x) := f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$ for $x \in I := (x_0, x_1)$.

replace by $\begin{cases} \geq \\ < \\ > \end{cases}$ for $\begin{cases} \text{concave} \\ \text{strictly convex} \\ \text{strictly concave} \end{cases}$



Actually, if we write $x = x_\alpha = \alpha x_1 + (1-\alpha)x_0$, then $\ell_I(x) = \alpha f(x_1) + (1-\alpha) f(x_0)$, so that $f(x) \leq \ell_I(x)$ becomes

$$(1-\alpha)(f(x) - f(x_0)) \leq \alpha(f(x_1) - f(x)).$$

Theorem 3: Assume f satisfies (*) and f'' exists on (a,b). Then

$f'' \begin{cases} \geq 0 \\ \leq 0 \end{cases}$ on $(a,b) \Rightarrow f$ is $\begin{cases} \text{convex} \\ \text{concave} \end{cases}$ on $[a,b]$.

Proof: By Corollary 2, $f'' \geq 0 \Rightarrow f'$ increasing

By MVT, $\exists\ c_0 \in (x_0, x)$ & $c_1 \in (x, x_1)$ s.t.

$$\begin{cases} f(x) - f(x_0) = f'(c_0)(x - x_0) = \alpha f'(c_0)(x_1 - x_0) \leq \alpha f'(c_1)(x_1 - x_0) \\ f(x_1) - f(x) = f'(c_1)(x_1 - x) = (1-\alpha) f'(c_1)(x_1 - x_0). \end{cases}$$

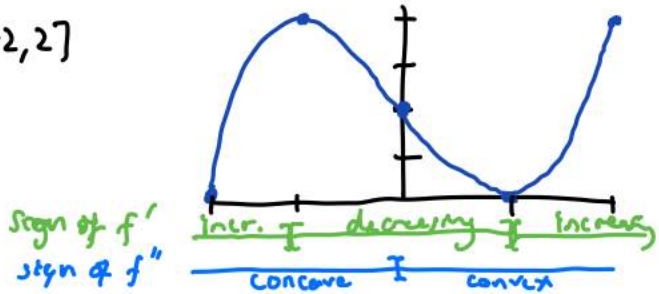
So $(1-\alpha)(f(x) - f(x_0)) \leq \alpha(1-\alpha) f'(c_1)(x_1 - x_0) = \alpha(f(x_1) - f(x))$. □

Ex 5/ Sketch $f(x) = (x-1)^2(x+2)$ on $[-2,2]$

$f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$

$f''(x) = 6x$

$f(-1) = 4$, $f(1) = 2$, $f(2) = 4$



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