Lecture 17: The Fundamental Theorems

Main Point of today's lecture:

- taking the "slope" function \( f(x) \) and \( f'(x) \)
- taking the "area-up-to-x" function \( f(x) \) and \( \int_c^x f(t) \, dt \)

are inverse operations.

Let \( I \) be an open interval.

**Definition:** \( P : I \to \mathbb{R} \) is a primitive (or antiderivative) of \( f : I \to \mathbb{R} \) if \( P' = f \) on \( I \).

**FTC I:** ("Indefinite integrals yield primitives")

Assume: \( f \) integrable on \([a,x]\) \( \forall x \in [a,b] \), and \( c \in [a,b] \).
(e.g. \( f \) is bounded + piecewise monotonic on \([a,b]\),
or \( f \) is piecewise continuous on \([a,b]\))

Set \( A(x) := \int_c^x f(t) \, dt \), \( \mathcal{D}_f := \{ x \in (a,b) | f \text{ is continuous at } x \} \).

Then: \( A'(x) \) exists and equals \( f(x) \) at every \( x \in \mathcal{D}_f \).

**FTC II.** Assume: \( f \) continuous on \( I \), and \( c \in [a,b] \). Then:

\[ \begin{array}{ll}
V.1 & ("Every primitive is an indefinite integral") \\
V.2 & ("Definite integral = change in primitive") \\
\end{array} \]

\[ P(x) = P(c) + \int_c^x f(t) \, dt \iff \int_a^b f(t) \, dt = P(b) - P(a) = P(x) \bigg|_a^b. \]
You've probably all had enough of using FTC II to compute integrals, but here is one we weren't able to do from the definition of the integral: fractional powers.

\[ \text{Ex 1: We showed } \frac{d}{dx} x^{q+1} = (q+1) x^q \text{ for any } q \neq 0. \text{ So } \frac{x^{q+1}}{q+1} \text{ is a primitive for } x^q \text{ so long as } q \neq -1, \text{ and (for } 0 \leq x \leq b) \]
\[ \int_a^b x^q \, dx = \frac{b^{q+1} - a^{q+1}}{q+1} \text{ by FTC II.} \]

\[ \text{Ex 2: If } q = -1, \text{ define (for } x > 0) \log(x) := \int_1^x \frac{1}{t} \, dt. \]
By FTC I, \[ \frac{d}{dx} \log(x) = \frac{1}{x}. \]

\[ \text{Ex 3: The FTC's elucidate the relationship between the MVTs. Suppose we only knew the one for derivatives, and let } f \text{ be continuous on } [a,b]. \text{ Then } f \text{ is integrable there and } \]
\[ \int_c^x f(t) \, dt = F(x) \]
\[ \left\{ \begin{array}{l}
\text{continuous on } [a,b] \text{ (by Lecture 12)} \\
\text{differentiable on } (a,b), \text{ with } F' = f \text{ (by FTC I)}. 
\end{array} \right. \]
So \( F \) satisfies the hypotheses of the MVT for derivatives, and \( F \) is continuous on \([a,b]\) and \( F(x) = \frac{1}{b-a} (F(b) - F(a)) \). Since \( F' = f \) and \( F(b) - F(a) = \int_a^b f(x) \, dx \) (FTC II), this yields \( f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx \).

If \( g \) is also continuous on \([a,b]\), with primitive \( G \), and \( g \) has primitive \( H \), then applying the Cauchy MVT to \( H \circ G \) gives \[ \frac{H'(c)}{G'(c)} = \frac{H(b) - H(a)}{G(b) - G(a)} \text{ hence } f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx \]

which is the Weighted MVT for Integrals!
Ex 4/ \[ p(t) = (\text{vertical}) \text{ position of ball} \]
\[ v(t) = p'(t) = \text{velocity} \]
\[ a(t) = v'(t) = p''(t) = \text{acceleration} \]

If \( a = a_0 \) is constant, then \( v(t) = a_0 t + v_0 \ (v_0 = v(0)) \)
and the change in position is \( p(T) - p(0) = \int_0^T (a_0 t + v_0) \, dt = a_0 \frac{T^2}{2} + v_0 T \).

For instance, if the ball starts from rest \( (v_0 = 0) \), and \( a_0 = -g \),
then \( p(T) - p(0) = -\frac{g}{2} T^2 \).

But if we factor in air-resistance, instead of \( p'' = -g \)
we have to “integrate” \( p'' = -g - k p' \), which is a full-fledged differential equation. (Can’t solve this yet!) //

Proof. The key point is that the rate of accumulation
of area under the graph of a function is proportional to its height.

Set \( A(x) = \int_c^x f(t) \, dt \).

Heuristically \[
\frac{\Delta A}{\Delta x} \sim \frac{f(x) \cdot \Delta x}{\Delta x} = f(x).\]

All we need to do is to make this a bit more precise.
Proof of FTC I: We must show that $A'(x)$ exists if $f$ is continuous at $x$.

Compute $A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h \to 0} \frac{\int_x^{x+h} f(t) \, dt}{h}$

$$= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t) \, dt.$$ 

To show this limit exists, let $\epsilon > 0$ : by continuity of $f$ at $x$, \( \exists \delta > 0 \) s.t.

\[ t \in (x-\delta, x+\delta) \implies f(t) - \frac{\epsilon}{2} < f(t) < f(t) + \frac{\epsilon}{2}. \tag{2} \]

Taking $0 < h < \delta$, $[x, x+h] \subset (x-\delta, x+\delta)$ and so $(2)$ holds on the interval of integration. By the comparison property,

$$\frac{1}{h} \int_x^{x+h} (f(t) - \frac{\epsilon}{2}) \, dt \leq \frac{1}{h} \int_x^{x+h} f(t) \, dt \leq \frac{1}{h} \int_x^{x+h} (f(t) + \frac{\epsilon}{2}) \, dt$$

\[
\therefore \quad f(x) - \frac{\epsilon}{2} \leq f(x) \leq f(x) + \frac{\epsilon}{2}.
\]

hence $\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t) \, dt - f(x) \leq \frac{\epsilon}{2} < \epsilon$. This gives $\lim_{h \to 0^+} \frac{1}{h} \int_x^{x+h} f(t) \, dt = f(x)$, and a similar analysis (reversing inequalities) gives the $\lim_{h \to 0^-}$.

Proof of FTC II: Define $A(x) := \int_x^x f(t) \, dt$. By FTC I and continuity of $f$, $A'(x) = f(x)$. So $P, A$ are both primitives \( \Rightarrow (P-A)' = P' - A' = f - f = 0 \Rightarrow P - A = k \) (constant)

\[ \therefore \quad k = P(c) - A(c) = P(c) \Rightarrow P(x) = P(c) + A(x). \]

\[ \text{evate at } x = c \]

\[ \Box \]