Lecture 18: Chain rule & substitution

Suppose that \( y = F(u) \) and \( u = g(x) \) determine a composite function \((F \circ g)(x) := F(g(x)) \) on an open interval \( I \). i.e. for those \( x \in I \) at which...

Theorem 1 (Chain rule): Wherever \( F'(g(x)) \) and \( g'(x) \) both exist,

\[(F \circ g)'(x) \text{ exists and equals } F'(g(x)) \cdot g'(x).
\]

Ways to think about this:

- as functions: \((F \circ g)' = (F' \circ g) \cdot g'\)
- in Leibniz notation: \( \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \)
- in words: when you compose functions, their rates of change multiply.

Proof: Fix an \( x \in I \) s.t. the hypothesis of the Thm. holds.

Define (for \( t \in \mathbb{R}(0) \))

\[
g(t) := \begin{cases} 
\frac{F(g(x)+t) - F(g(x))}{t}, & \text{if } t \neq 0 \\
F'(g(x)), & \text{if } t = 0 
\end{cases}
\]

which is continuous at 0 since \( F \) is differentiable at \( g(x) \). Clearly

\[
(t \ x \ t \ g(t) = F(g(x)+t) - F(g(x)) 
\]

holds in \( \mathbb{R}(0) \).

Note that \( k(h) = g(x+h) - g(x) \) is continuous at
\[ h = 0 \text{ (with value 0)} \text{ since } g \text{ is differentiable at } x. \text{ So for } h \text{ in a smaller \textit{nbhd.} } N_1(0) \text{ we have } \frac{h}{h} \in N(0), \text{ and substituting } t = k(h) \text{ in } (x) \text{ and dividing by } h \text{ gives}
\]
\[ \frac{k(h)}{h} q(k(h)) = \frac{F(g(x) + k(h)) - F(g(x))}{h} \]

for \( h \in N^*_1(0) \) (punctured \textit{nbhd.}), i.e.
\[ \frac{g(x+h)-g(x)}{h} \cdot q(k(h)) = \frac{F(g(x+h)) - F(g(x))}{h}. \]

Taking \( h \to 0 \) limits on both sides gives the result, since
\[ \lim_{h \to 0} q(k(h)) = q(\lim_{h \to 0} h(h)) = q(0) = F'(g(x)). \]

\[ \boxed{\text{Ex 1} \quad f(x) = \frac{1}{\sqrt{1 + x^2}} \Rightarrow f'(x) = \frac{1}{2\sqrt{1+x^2}} \cdot 2x} \]

\[ \boxed{\text{Ex 2} \quad \frac{d}{dx} \int_0^{g(x)} f(t) \, dt = f(g(x)) \cdot g'(x) \cdot \cos(x)} \]

\[ \boxed{\text{Ex 3} \quad f(x) = \sin(\sin(\sin(x))) \Rightarrow f'(x) = \frac{du}{dx} = \frac{dy}{dy} \cdot \frac{dx}{dv} \cdot \frac{dv}{dx}} \]

\[ \boxed{\text{Ex 4} \quad x^2 + y^2 = r^2 \text{ gives } y \text{ implicitly rather than explicitly } (= f(x)) \text{ in terms of } x. \text{ Since it is secretly a function of } x, \text{ you must use the chain rule and write } \frac{dy^2}{dx} = \frac{dy}{dx} \cdot \frac{dy}{dy} = 2y \cdot y', \text{ hence } 2x + 2yy' = 0 \Rightarrow y' = -\frac{x}{y} \Rightarrow \text{tangent to a circle is perpendicular to the radius (why?).} } \]
Ex 5/(More implicit diff.)

\[ x^3 + y^3 = 1 \rightarrow \frac{d}{dx} x^3 + \frac{d}{dx} y^3 = 0 \rightarrow 3x^2 + 3y^2 y' = 0 \]

\[ \Rightarrow 2x y^3 + 2(y^2 y')^2 + y^5 y'' = 0 \rightarrow 2x (y^3 + y')^2 + y^5 y'' = 0 \]

\[ \Rightarrow y'' = -2x y^{-5}. \]

Ex 6/(A related rates problem)

![Diagram of a backyard pool]

(V = volume)

\[ r(h) = \sqrt{10^2 - (10-h)^2} = \sqrt{20h - h^2} \]

- Calculate \( \frac{dV}{dh} \) when \( h = 5 \):

\[ V = \int_0^h A(h) \, dh \]

\[ \frac{dV}{dh} = A(h) = \pi(r(h))^2 = \pi(20h - h^2) \]

\[ \frac{dV}{dh} \bigg|_{h=5} = \pi(100 - 25) = 75\pi \text{ ft}^3 / \text{ft} \]

- If \( \frac{dV}{dt} = 5\sqrt{3} \) (constant), find \( \frac{dr}{dt} \) when \( h = 5 \).

\[ \frac{dr}{dt} = \frac{dr}{dh} \cdot \frac{dh}{dt} \cdot \frac{dV}{dh} = \frac{r'(h) \cdot 5\sqrt{3}}{\sqrt{20h - h^2}} \text{, where } r'(h) = \frac{10-h}{\sqrt{20h - h^2}}. \]

\[ r'(5) = \frac{5}{\sqrt{75}} = \frac{1}{\sqrt{3}} \text{ and } \frac{dr}{dt} \bigg|_{h=5} = \frac{\sqrt{3} \cdot 5\sqrt{3}}{75\pi} = \frac{1}{15\pi} \text{ ft} / \text{s}. \]

Now, the entire reason I waited until now to do the chain rule was so that we could also do the “inverse of the chain rule” for integrals, otherwise known as substitution.
Theorem 2: If $g'$ is continuous on $I$, $f$ is continuous on $g(I)$, and $x, c \in I$, then

$$
\bar{F}(x) := \int_c^x f(g(t)) \cdot g'(t) \, dt = \int_{g(c)}^{g(x)} f(u) \, du.
$$

Proof: Set $F(w) := \int_g^w f(u) \, du$, so that $F'(w) = f(w)$.

By the Chain rule, $F'(x) = f(g(x)) \cdot g'(x) = F'(g(x)) \cdot g'(x) = (F \circ g)'(x)$.

Hence $F' - F \circ g = K$ (constant), and evaluating at $c$ gives

$$
K = \bar{F}(c) - F(g(c)) = 0 - 0 = 0. \quad \text{So } \bar{F}(x) = F(g(x)). \quad \square
$$

Example:

$$
\int_0^{\sqrt[3]{\frac{1}{3}}} \frac{\sin(x)}{\sqrt{\cos^3(x)}} \, dx = \int_{\cos(0)}^{\cos(\pi/3)} \frac{\sin(u)}{\sqrt{u^3}} \, du = -\int_1^{\sqrt[3]{\frac{1}{3}}} u^{-3/2} \, du
$$

Let $u = \cos(x)$

$du = -\sin(x) \, dx$

$$
= -\left[ u^{-3/2 + 1} \right]^{1/2}_{-3/2 + 1} = 2 u^{-1/2} \bigg|_{1}^{1/2} = 2 \sqrt{2} - 2. \quad \square
$$