Lecture 2: Area of a parabolic segment

Around a quarter of a millennium B.C., Archimedes managed to compute what we would now recognize as a definite integral. We begin with Apostol’s “modern” version of his calculation, which is really only a special case.

For any positive integer \( n \), we may consider the areas of the “unions of boxes” above and below the curve \( y = x^2 \):

\[
S_n = \sum_{k=1}^{n} \frac{b}{n} \left( \frac{kb}{n} \right)^2 = \frac{b^3}{n^3} \sum_{k=1}^{n} k^2 = \frac{b^3}{n^3} P_2(n)
\]

and

\[
s_n = \sum_{k=0}^{n-1} \frac{b}{n} \left( \frac{kb}{n} \right)^2 = \frac{b^3}{n^3} \sum_{k=0}^{n-1} k^2 = \frac{b^3}{n^3} P_2(n-1)
\]

Our intuition regarding areas (when one region contains another) dictates that

\[
s_n \leq A \leq S_n \quad \text{for all } n \geq 1.
\]
Now (**) from Lecture 1 says that
\[ P_2(n-1) < \frac{n^3}{3} < P_2(n) \quad (\text{for all } n \geq 1), \]
or (multiplying by \( \frac{n^3}{3} \))
\[ \left( s_n = \right) \frac{b^3}{n^3} P_2(n-1) < \frac{b^3}{3} < \frac{b^3}{n^3} P_2(n) \quad ( = S_n ). \]
Suppose \( A \neq \frac{b^3}{3} \), so that \( \varepsilon := \left| A - \frac{b^3}{3} \right| > \frac{1}{N} \) for some positive integer \( N \). Since both \( A \) and \( \frac{b^3}{3} \) are sandwiched between \( s_n \) and \( S_n \), we get
\[ \frac{1}{N} < \varepsilon < S_n - s_n = \frac{b^3}{n^3} \left( P_2(n) - P_2(n-1) \right) = \frac{b^3}{n^3} \]
for all \( n \). But this is false as soon as \( n \geq \frac{b^3}{N} \). So \( A = \frac{b^3}{3} \).

[Remark: If we had defined limits, we could just plug in our formulas for \( P_2(n-1) \) and \( P_2(n) \) and use the squeeze theorem. But we don't yet have the concept of "limit". (Sorry.)]

But this isn't remotely what Archimedes did. Let \( A \) be the area of the parabolic segment bounded by a line \( l \) and a parabola \( P \):

\[ \text{The original source:} \]
http://www.math.ubc.ca/~cass/archimedes/parabola.html
and $A'$ the area of the triangle $APB$ (where $P$ is the point on $P$ furthest from $l$):

Archimedes’s Theorem: $A = \frac{4}{3} A'$.

This is much more general than, but immediately implies, the result in Apostol: for consider the picture

We have \( \text{area (box)} = A + 2A \)

\[ 2b \cdot b^2 = \frac{4}{3} \cdot \text{area (APB)} + 2A \]

\[ 2b^3 = \frac{4}{3} b^3 + 2A \]

\[ \Rightarrow A = \frac{1}{3} b^3 \text{, as desired.} \]

So how did Archimedes prove this more general theorem?
Illustration of the "method of exhaustion".

We shall use several results known in Archimedes' time without proof: First, we have a parallelogram (with area $2a_0$) in which $M$ is the midpoint of $AB$ and $DC$ is tangent to $Q$ at $P$:

$$A < 2a_0 \implies \frac{A}{2} < a_0 \implies e_0 = A - a_0 < \frac{1}{2} A.$$
$S_i$ is the parabolic segment approximated by $S_i$, then we deduce $S_i - S_i < \frac{1}{2} S_i$, and thus

$$e_i = A - A_1 = (A_0 + S_i + S_{i+1}) - (A_0 + S_i + S_{i+1}) = (S_i - S_i) + (S_{i+1} - S_{i+1}) < \frac{1}{2} (S_i + S_{i+1}) = \frac{1}{2} e_0.$$  
Continuing in this fashion gives $e_j < \frac{1}{2} e_{j-1}$ (for each $j$)

hence $e_j < \frac{1}{2^j} M A$. (So these "errors" go to zero.)

Next, we claim that

$$(1) \quad S_i + S_{i+1} = \frac{1}{2} A_0,$$

so that $A_1 = A_0 + \frac{1}{2} A_0$, and (continuing the process)

$$A_2 = A_0 + \frac{1}{4} A_0 + \frac{1}{4^2} A_0, \quad \text{(why ?)}$$

$$A_n = A_0 + \frac{1}{4} A_0 + \cdots + \frac{1}{4^n} A_0.$$

Another key property known by the ancient Greeks is that

$$\frac{PA}{PM} = \left(\frac{QP_2}{Q^2}\right)^2 = \frac{MN^2}{(2 MN)^2} = \frac{1}{4}$$

(basically says that the parabola still looks like $y = x^2$ in the oblique coordinate system with axes given by the parallelogram).
So we have \( PM = 4PQ \), which implies \( P_2N = 3PQ \), hence \( S_{RN} = \frac{1}{2} PM = 2PQ \)  

\[ P_2R = P_2N - RN = 3PQ - 2PQ = PQ \]

\[ \Rightarrow S_2 = \text{area}(PP_2B) = \frac{1}{2} \text{area}(PNB) = \frac{1}{4} \text{area}(PMB). \]

Similarly, \( S_1 = \frac{1}{4} \text{area}(PMA) \), and so \( S_1 + S_2 = \frac{1}{4} \text{area}(PAB) = \frac{1}{4} A_0 \), as desired.

Finally, we want to show \( A = \frac{4}{3} A_0 \).

We know that \( 0 \leq A - A_n < \frac{1}{2^{n+1}} A \) and that

\[ A_n = \left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^n}\right) A_0 = S_{\frac{1}{4}}(n) A_0 = \frac{4}{3} A_0 - \frac{1}{3 \cdot 4^n} A_0. \]

That is,

\[ 0 \leq A - \frac{4}{3} A_0 + \frac{1}{3 \cdot 4^n} A_0 < \frac{1}{2^{n+1}} A \]

\[ \Rightarrow -\frac{1}{3 \cdot 4^n} A_0 \leq A - \frac{4}{3} A_0 < \frac{1}{2^{n+1}} A - \frac{1}{3 \cdot 4^n} A_0. \]

If the middle quantity were positive, you'd get a contradiction to the "<" by taking \( n \) large enough. Were it negative, you'd get a contradiction to the "\( \leq \)."

So \( A - \frac{4}{3} A_0 = 0 \), done. (Again, it's all much easier once we have limits, which Archimedes didn't:

\[ A = \lim A_n = \lim \frac{1}{4} P_{\frac{1}{4}}(n) A_0 = \lim \frac{1}{1 - \frac{1}{4^{n+1}}} A_0 = \frac{1}{3/4} A_0 = \frac{4}{3} A_0. \]

Well! That was hard. The rest of the week will be back to fundamentals — sets, real numbers, and induction.