Lecture 2: Area of a parabolic segment

Around a quarter of a millennium B.C., Archimedes managed to compute what we would now recognize as a definite integral. We begin with Apostol’s “modern” version of his calculation, which is really only a special case.

For any positive integer \( n \), we may consider the areas of the “unions of boxes” above and below the curve \( y = x^2 \):

\[
S_n = \sum_{k=1}^{n} \frac{b}{n} \left( \frac{kb}{n} \right)^2 = \frac{b^3}{n^3} \sum_{k=1}^{n} k^2 = \frac{b^3}{n^3} P_2(n)
\]

and

\[
s_n = \sum_{k=0}^{n-1} \frac{b}{n} \left( \frac{kb}{n} \right)^2 = \frac{b^3}{n^3} \sum_{k=1}^{n-1} k^2 = \frac{b^3}{n^3} P_2(n-1)
\]

Our intuition regarding areas (when one region contains another) dictates that

\[ s_n \leq A \leq S_n \quad \text{for all } n \geq 1. \]
Now (***) from Lecture 1 says that
\[ P_2(n-1) < \frac{n^3}{3} < P_2(n) \quad \text{(for all } n \geq 1), \]
or (multiplying by \( \frac{b^3}{n^3} \))
\[ \left( s_n = \right) \frac{b^3}{n^3} P_2(n-1) < \frac{b^3}{3} < \frac{b^3}{n^3} P_2(n) \quad \left( = S_n \right). \]

Suppose \( A \neq \frac{b^3}{3} \), so that \( \varepsilon := \left| A - \frac{b^3}{3} \right| > \frac{1}{N} \) for some positive integer \( N \). Since both \( A \) and \( \frac{b^3}{N} \) are sandwiched between \( s_n \) and \( S_n \), we get
\[ \frac{1}{N} < \varepsilon < S_n - s_n = \frac{b^3}{n^3} \left( P_2(n) - P_2(n-1) \right) = \frac{b^3}{n^3} \]
for all \( n \). But this is false as soon as \( n \geq b^2 N \). So \( A = \frac{b^3}{3} \).

\[ \text{[Remark: If we had defined limits, we could just plug in our formulas for } P_2(n-1) \text{ and } P_2(n) \text{ and use the squeeze theorem. But we don't yet have the concept of "limit". (Sorry.)]} \]

But this isn't remotely what Archimedes did.
Let \( A \) be the area of the parabolic segment bounded by a line \( l \) and a parabola \( L \):

\[ \text{The original source:} \]
http://www.math.ubc.ca/~cass/courses/m309-8a/java/images/archimedes/parabola.html
and $A'$ the area of the triangle $APB$ (where $P$ is the point on $P$ furthest from $l$):

**Archimedes's Theorem:** $A = \frac{4}{3} A'$.

This is much more general than, but immediately implies, the result in Apostol: for consider the picture.

We have \[ \text{area (box)} = A + 2A \]

\[ 2b \cdot b^2 = \frac{4}{3} \cdot \text{area (APB)} + 2A \]

\[ 2b^3 = \frac{4}{3} b^3 + 2A \]

\[ \Rightarrow A = \frac{1}{3} b^3, \text{ as desired.} \]

So how did Archimedes prove this more general theorem?
Illustration of the "method of exhaustion"

Write

\[ e_j := A - A_j \]

for the "leftover" area at each step.

We shall use several results known in Archimedes' time without proof: First, we have a parallelogram (with area \(2a_0\)) in which \(M\) is the midpoint of \(AB\) and \(DC\) is tangent to \(Q\) at \(P\):

\[ A < 2a_0 \Rightarrow \frac{A}{2} < a_0 \]

\[ \Rightarrow \varepsilon_0 = A - a_0 < \frac{1}{2} A. \]

By the same token, if
$\delta_i$ is the parabolic segment approximated by $S_i$; then we deduce $\delta_i - S_i < \frac{1}{2} \delta_i$, and thus

$$\varepsilon_i = A_i - A_1 = (A_0 + S_1 + S_2) - (A_0 + S_1 + S_2) = (S_1 - S_1) + (S_2 - S_1) < \frac{1}{2}(S_1 + S_2) = \frac{1}{2} \varepsilon_0.$$  
Continuing in this fashion gives $\varepsilon_j < \frac{1}{2} \varepsilon_{j-1}$ (for each $j$) hence $\varepsilon_j < \frac{1}{2^j} A_1$. (So these “errors” go to zero.)

Next, we claim that

(1) \[ S_1 + S_2 = \frac{1}{4} A_0, \]

so that $A_1 = A_0 + \frac{1}{4} A_0$, and (continuing the process)

\[
A_2 = A_0 + \frac{1}{4} A_0 + \frac{1}{4^2} A_0 \]

\[
A_3 = A_0 + \frac{1}{4} A_0 + \cdots + \frac{1}{4^3} A_0.
\]

Another key property known by the ancient Greeks is that

\[
\frac{PA}{PM} = \frac{(OP_2)^2}{MB^2} = \frac{MN^2}{(2MN)^2} = \frac{1}{4}
\]

(basically says that the parabola still looks like $y = x^2$ in the oblique coordinate system with axes given by the parallelogram).
So we have \( PM = 4PA \), which implies \( P_2N = 3PA \), hence
\[
\begin{align*}
S_{RN} &= \frac{1}{2} PM = 2PA \\
\Rightarrow P_2R &= \frac{1}{2} RN \\
P_2R &= P_2N - RN = 3PA - 2PA = PA
\end{align*}
\]
\[
\Rightarrow S_2 = \text{area}(PP_2B) = \frac{1}{4} \text{area}(PNB) = \frac{1}{4} \text{area}(PMB).
\]
Similarly, \( S_1 = \frac{1}{4} \text{area}(PMA) \), and so
\[
S_1 + S_2 = \frac{1}{4} \text{area}(PAB) = \frac{1}{4} A_0,
\]
as desired.

Finally, we want to show \( A = \frac{4}{3} A_0 \).

We know that \( 0 \leq A - A_0 < \frac{1}{2^{n+1}} A \) and that
\[
A_0 = \left( 1 + \frac{1}{4} + \frac{1}{4^2} + \ldots + \frac{1}{4^n} \right) A_0 = \sum_{n=0}^{\infty} \frac{1}{4^n} A_0 = \frac{4}{3} A_0 - \frac{1}{3} \cdot \frac{1}{4^n} A_0.
\]
That is,
\[
0 \leq A - \frac{4}{3} A_0 + \frac{1}{3} \cdot \frac{1}{4^n} A_0 < \frac{1}{2^{n+1}} A
\]
\[
\Rightarrow -\frac{1}{3} \cdot \frac{1}{4^n} A_0 \leq A - \frac{4}{3} A_0 < \frac{1}{2^{n+1}} A - \frac{1}{3} \cdot \frac{1}{4^n} A_0.
\]
If the middle quantity were positive,
you'd get a contradiction to the "<" by taking \( n \) large enough. Were it negative, you'd get a contradiction to the "\( \leq \)".

So \( A - \frac{4}{3} A_0 = 0 \), done. (Again, it's all much easier once we have limits, which Archimedes didn't.)
\[
A = \lim_{n \to \infty} A_n = \lim_{n \to \infty} P_1(n) A_0 = \lim_{n \to \infty} \frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{4}} A_0 = \frac{1}{3/4} A_0 = \frac{4}{3} A_0.
\]

Well! That was hard. The rest of the week will be back to foundations — sets, real numbers, and induction.