Lecture 20: The logarithm

**Definition:** \( \log(x) := \int_1^x \frac{dt}{t} \) for \( x \in \mathbb{R}^+ \).

Let's deduce some properties from this:

1. \( \log(1) = 0 \)
2. \( \log \) is differentiable \& (twice) continuous on \( \mathbb{R}^+ \), by FTC
3. \( \log'(x) = \frac{1}{x} \)
4. \( \log(wx) = \log(w) + \log(x) \) \( \forall x, w \in \mathbb{R}^+ \)
   
   **Proof:** \( \int_1^w \frac{dx}{x} = \int_1^x \frac{dt}{t} + \int_x^w \frac{dx}{x} = \log(w) + \int_1^x \frac{du}{u} = \log(w) + \log(x). \) \( \square \)
5. \( \log \left( \frac{1}{w} \right) = -\log(w) \) (take \( x = \frac{1}{w} \) in (4))
6. \( \log(w^r) = r \log(w) \) for any \( r \in \mathbb{Q} \)
   
   **Proof:** Inductively one has for \( n \in \mathbb{N} \) \( \log(z^n) = n \log(z) \). Taking \( z = w^m \Rightarrow \frac{1}{m} \log(w) = \log(w^m) \Rightarrow \log(w^{ mn}) = \frac{mn}{m} \log(w) \). Finally combine with (5). \( \square \)
7. \( \log \) is strictly increasing (since \( \log' > 0 \))
8. \( \log \) is strictly concave (since \( \log'' < 0 \))
9. \( \log \) is neither bounded above nor bounded below
   
   **Proof:** Given \( M \in \mathbb{R}^+ \), Archimedean property \( \Rightarrow \) \( \exists n \in \mathbb{N} \) s.t. \( n \log 2 > M \)
   
   \( \Rightarrow \log 2^n > M \) \& \( \log 2^{-n} < -M \). \( \square \)
(10) \( \log \) has a strictly increasing, continuous, inverse defined on all of \( \mathbb{R} \); call this \( \exp \). Define \( e := \exp(1) \in \mathbb{R}^+ \).

**Proof:** Given \( y_0 \in \mathbb{R} \), choose (by (9)) \( n \in \mathbb{N} \) s.t. \( y_0 \in [\log(2^n), \log(2^{n+1})] \). By the Intermediate Value Thm., \( \exists \ x_0 \in [2^n, 2^{n+1}] \) s.t. \( \log(x_0) = y_0 \).

If \( \log(x_i) = y_0 \) also, then \( x_i \geq x_0 \) \& \( x_i < x_0 \) both give contradictions (use (7)); so \( x_0 \) is unique, and we can set \( \exp(y_0) = x_0 \).

For continuity, see the end of lecture 13. \( \square \)

(11) \( \log(e) = 1 \).

**Definition:** For any \( b \in \mathbb{R}^+ \), \( \log_b(x) := \frac{\log(x)}{\log(b)} \).

This has \( \log_b(1) = 1 \) and satisfies properties (1) - (10), except replacing (3) by \( \log_b(x) = \frac{1}{x \log(b)} \), and \( \log_b(1) = \frac{1}{\log(b)} \).

In particular,

It is interesting to ask whether the family of functions \( \{ \log_b \mid b \in \mathbb{R}^+ \} \) (and the zero-function) are the only functions defined on all of \( \mathbb{R}^+ \) which satisfy the functional equation

(12) \( \log(bx) = \log(b) + \log(x) \).

Actually, the answer is "no": there are uncountably many pathological, everywhere-discontinuous functions satisfying (12). But if we limit ourselves to continuous functions, or monotonic functions, the answer is "yes":

\( \frac{\log(x)}{\log(b)} \) for any \( b \in \mathbb{R}^+ \) s.t. \( b \neq 1 \).
Theorem: Suppose \( f: \mathbb{R}^+ \to \mathbb{R} \) satisfies (\( \ast \)), \( f \) is either strictly monotonic or continuous. Then \( f \) is either 0 or log, for some \( b \in \mathbb{R}^+ \).

Proof: To begin, notice that (\( \ast \)) \( \Rightarrow f(1) = f(1) + f(0) \Rightarrow f(0) = 0 \), as well as properties (\( \delta \)) - (\( \epsilon \)).

First, suppose \( f \) is strictly monotonic. We claim that \( f \) is continuous. We may reduce \( f \) by \(-f\), if necessary, to make \( f \) strictly increasing. It suffices to show \( f \) continuous at 1, since then \( \lim_{x \to 1} f(x) = \lim_{x \to x_0} f(x) + f(x_0) = \lim_{x \to x_0} f(x) + f(x_0) = f(x) + f(x_0) = f(x_0) \). Let \( \epsilon > 0 \) be given, and take \( n \in \mathbb{N} \), \( 1 \leq \frac{f(n)}{\epsilon} \), \( \delta := \min \{2^{-k}, 1-2^{-k} \} (\epsilon) \). Then

\[
1 - \delta < x < 1 + \delta \Rightarrow 2^{-k} < x < 2^{-k} \Rightarrow -\frac{1}{n} f(x) < f(x) < \frac{1}{n} f(x) \quad \text{f str. inc.}
\]

\[
-\epsilon < f(x) < \epsilon \quad \text{so} \quad \lim_{x \to 1} f(x) = 0 = f(c) \quad \text{done.}
\]

Thus it will suffice to prove the result assuming \( f \) continuous, and not identically zero. Then we have \( f(x^n) > 0 \) for some \( x \in \mathbb{R}^+ \), hence \( f(x^n) = N f(x) > 1 \) for some \( N \in \mathbb{N} \) (Arch. prop.), while \( f(x^n) < 0 \). By IVT, \( \exists b \in \mathbb{R}^+ \) s.t. \( f(b) = 1 \), and \( F := f - \log \) is a continuous function satisfying (\( \ast \)) which is 0 at \( b \) hence of any rational power of \( b \). Given \( x < y \), we know that between \( \log, x \) and \( \log, y \) there is some \( q \in \mathbb{Q} \); and then \( b^q \in (x, y) \) (otherwise strict monotonicity of \( \log \) yields a contradiction).

So if \( F(c) \neq 0 \) for some \( c \in \mathbb{R}^+ \), \( \exists \delta > 0 \) s.t. \( x \in (c - \delta, c + \delta) \)

\[
\Rightarrow |F(x) - F(c)| < \frac{\epsilon}{\delta} \Rightarrow F(x) \neq 0 \quad \text{This is a contradiction since there is a “b2” in (c - \delta, c + \delta). So F = 0. D}
\]
This is a stronger result than the book's, which assumes \( f \) is differentiable to get \( f'(wxt) = f(w) + f'(x) \Rightarrow \frac{w f'(wxt)}{x} = f'(x) \Rightarrow f'(w) = \frac{f'(1)}{w} \Rightarrow f(w) = \log_b(w) \) where \( f'(1) = \frac{1}{\log_b(e)} \).

(i.e. \( b = \exp \left( \frac{1}{f'(1)} \right) \)).

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**Two quick applications**

1. \( \int f'(wxt) \, dx = \log(w) + C \) and \( f(w) > 0 \). If not, use instead: \( \int \log(x) \, dx = \begin{cases} \log x, & x > 0 \\ \log(-x), & x < 0 \end{cases} \).

   Since \( \int \log(x) \, dx = \frac{x \log x}{x} + C = \frac{1}{x} \), this makes the u-sub. above valid on any interval when \( u \neq 0 \).

   \[ \text{Ex.} \quad \int \tan(x) \, dx = \int \frac{\sin(x)}{\cos(x)} \, dx = -\int \frac{\sin u}{u} \, du = -\log|u| + C = -\log|\cos x| + C \]

2. **Logarithmic differentiation**: \( g = L \cdot f \Rightarrow g' = (L' \cdot f + f') \cdot \frac{g}{f} \Rightarrow f' = f \cdot g' \).

   \[ \text{Ex.} \quad f(x) = \frac{x^2}{(1 + x^4)^2} \]

   \[ g(x) = (L \cdot f)(x) = 2 \log|x| + \log|\cos x| - \frac{7 \log(1 + x^4)}{x^2} \]

   \[ g'(x) = \frac{2 x \cos x}{\cos x} - \frac{28 x^3}{1 + x^4} \]

   \[ f'(x) = f'g' = \frac{2 x \cos x}{(1 + x^4)^2} - \frac{x^2 \sin x}{(1 + x^4)^2} - \frac{28 x^5 \cos x}{(1 + x^4)^3} \]

I should also point out that log gives a really nice approximation to \( \pi(x) = \# \text{ of primes } \leq x \). In fact, \( \pi(x) \approx \frac{x}{\log(x)} \) when "\( \approx \)" means "is asymptotic to", i.e. \( \lim_{x \to \infty} \frac{\pi(x)}{x/\log(x)} = 1 \). An even better approx. is \( \int_1^x \frac{dt}{\log(t)} \).