Theorem: Suppose \( f \) is strictly monotonic and continuous on an interval \( I \). Then it has a unique (strictly monotonic and continuous) inverse function \( g \). [We already knew this.] Moreover, if \( f \) is differentiable at \( x \), then \( g \) is differentiable at \( f(x) \), with \( g'(f(x)) = \frac{1}{f'(x)} \).

Remark: This is often written \( (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} \), by writing \( g = f^{-1} \) and substituting \( f^{-1}(y) \) for \( x \).

This looks really nontrivial. Or (thinking \( y = f(x) \), \( x = f^{-1}(y) \)) you can write it in the much more trivial-looking form

\[
\frac{dx}{dy} = \frac{1}{gy'/x'}.
\]

Proof: We argue exactly as in the proof for \( \exp \): write

\[
G(h) := g(f(x)+h) - g(f(x)) \quad \text{and} \quad F(u) := \frac{u}{f(x+u) - f(x)}.
\]

Then

\[
\lim_{h \to 0} F(u) = \lim_{h \to 0} \frac{G(h)}{u} = \lim_{h \to 0} \frac{g(f(x+h)) - g(f(x))}{h} = \frac{1}{f'(x)}, \quad \text{while}
\]

(by continuity and strict monotonicity of \( g \)) \( G \) is continuous at 0 with \( G(0) = 0 \) and \( G(h) \neq 0 \) for \( h \in \mathbb{N}^*(0) \). By the limit laws for compositions, \( \lim_{h \to 0} F(G(h)) = \lim_{h \to 0} F(u) = \frac{1}{f'(u)} \) on the one hand; while on the other

\[
\lim_{h \to 0} F(G(h)) = \lim_{h \to 0} \frac{G(h)}{h(x+G(h)) - f(x)} = \lim_{h \to 0} \frac{g(f(x+h)) - g(f(x))}{h} = g'(f(x)).
\]
\[
\text{Here we used that } f(x+g(f(x)+h)-g(f(x)+h)) = f(x) = f(g(f(x)+h)) - f(x) = f(x) + h - f(x) = h.
\]

\text{Ex/ Define } \arcsin(x) := \sin^{-1}(x), \text{ on the interval } [-1, 1].

On \((-1, 1), \arcsin'(x) = \frac{1}{\sin'(\arcsin(x))} = \frac{1}{\cos(\arcsin(x))}.

\text{Ex/ Define } \arctan(x) := \tan^{-1}(x), \text{ on } \mathbb{R}. \text{ We have}

\arctan'(x) = \frac{1}{\tan'(\arctan(x))} = \frac{1}{\sec(\arctan(x))} = \frac{1}{\sqrt{1+x^2}}.

\text{Ex/ } \int \frac{dx}{\sqrt{1-2x-x^2}} = \int \frac{dn}{\sqrt{2-(n+1)^2}} = \frac{1}{\sqrt{2}} \int \frac{dn}{\sqrt{1-(n+1)^2}}

= \int \frac{dn}{\sqrt{1-n^2}} = \arcsin(n) + C = \arcsin\left(\frac{x+1}{1}\right) + C
Hyperbolic functions

\[ \sinh(x) = \frac{e^x - e^{-x}}{2} \quad \cosh(x) = \frac{e^x + e^{-x}}{2} \]

We define \( \tanh = \frac{\sinh}{\cosh} \) and so forth, in analogy to trigonometric functions. (In fact, it's more than an analogy, as we'll see when we get to complex numbers.)

Why "hyperbolic"? Because they satisfy the equation \( x^2 - y^2 = 1 \) of a hyperbola:

\[ \cosh^2 x - \sinh^2 x = \frac{e^{2x} + 2e^x - e^{2x}}{4} - \frac{e^{2x} - 2e^x + e^{-2x}}{4} = 1. \]

We also have

- \( \sinh(-x) = \frac{e^{-x} - e^x}{2} = -\sinh(x), \cosh(-x) = \cosh(x) \)
- \( \frac{d}{dx} e^x = e^x \Rightarrow \sinh'(x) = \cosh(x), \cosh'(x) = \sinh(x) \); and
  \[ \tanh'(x) = \frac{\cosh(x)^2 - \sinh(x)^2}{\cosh(x)^2} = \frac{1}{\cosh(x)^2} : \cosh^2(x) \]
- \( \sinh(x) \cosh(y) + \cosh(x) \sinh(y) = (e^x e^y + e^{-x} e^{-y}) + (e^x e^{-x})(e^y e^{-y}) \)
  \[ = \frac{e^{xy} - e^{-xy} + e^{xy} + e^{-xy}}{4} \]
  \[ = \frac{e^{xy} - e^{-xy} + e^{xy} - e^{-xy}}{4} \]
  \[ = \sinh(xy) \]

and many other identities analogous to the trigonometric ones. We may even consider (on \( \mathbb{R} \))

\[ \left( \tanh^{-1}(x) \right)' = \frac{1}{\tanh'(\tanh^{-1}(x))} = \frac{1}{\cosh^2(\tanh^{-1}(x))} = \frac{1}{x^2} \]

which is very meaningful, given that

\[ \frac{1}{1 - x^2} = \frac{1}{(1-x)(1+x)} = \frac{1}{1-x} + \frac{1}{1+x} \]

\[ \Rightarrow \quad \frac{d}{dx} \left( \frac{1}{2} \log(1+x) - \frac{1}{2} \log(1-x) \right) = \frac{1}{1-x^2} \]

Since \( \tanh(0) = 0 = \frac{1}{2} \log(\frac{1+0}{1-0}) \), it implies that \( \tanh^{-1}(x) = \frac{1}{2} \log(\frac{1+x}{1-x}) \) on \((-1, 1)\). Can you prove this more directly?
Proposition: Given polynomials \( F(x) \) and \( G(x) \), there exist unique polynomials \( q(x) \) and \( r(x) \) such that

\[
F = qG + r \quad \text{and} \quad \deg(r) < \deg(G).
\]

Example:

\[
\begin{align*}
F &= x^3 + x + 3 \\
G &= x + 1
\end{align*}
\]

\[
\begin{array}{c}
x^2 - x + 2 \\
x + 1 \\
- (x^2 + x) \\
2x + 3 \\
\hline
\end{array}
\]

\[
q = x^2 - x + 2
\]

\[
r = 1
\]

(Of course "\( q \)" stands for quotient, and "\( r \)" remainder)

Proof: Let \( S := \{ F - pG \mid p \text{ polynomial} \} \), where \( G = a x^n + \text{lower-degree terms} \). Let \( r = b x^m + \text{lower-degree terms} \) be an element of \( S \) of lowest degree. If \( m \geq n \), then \( r - \frac{b}{a} x^{m-n} G \) is an element of \( S \) of degree less than \( m \), a contradiction. This proves existence.

For uniqueness, suppose \( F = qG + r = QG + R \) both satisfy (1). If \( q \neq Q \) then the left-hand side of

\[
(q - Q)G = R - r
\]

has strictly greater degree than \( R - r \), which is absurd. So \( q = Q \), hence \( R = r \).