Given a function \( f \) with continuous \((n+1)\)th derivative in a neighborhood \( N(a) = (a-c, a+c) \) of \( a \), we derived the formula

\[
E_n(x) := f(x) - T_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k
\]

\[
\equiv \int_{a}^{x} \frac{f^{(n+1)}(t)}{n!} (x-t)^n \, dt
\]

for the error in the \( n \)th Taylor approximation (polynomial) at \( a \).

This leads at once to the

\[
\text{for } x \in [a, a]
\]

**Theorem 1:** \( |E_n(x)| \leq \frac{\sup_{c \in (a, x]} |f^{(n+1)}(t)|}{(n+1)!} |x-a|^{n+1} \).

**Proof:** In general, we can bound integrals of continuous functions by reasoning that

\[
-F(t) \leq F(x) \leq F(c) \Rightarrow -\int_{a}^{b} F(t) \, dt \leq \int_{a}^{b} F(x) \, dt \leq \int_{a}^{b} F(t) \, dt
\]

\[
\Rightarrow \left| \int_{a}^{b} F(x) \, dx \right| \leq \int_{a}^{b} |F(x)| \, dx.
\]

Writing \( M := \sup_{t \in (a, x]} |f^{(n+1)}(t)| \), we have

\[
|E_n(x)| \leq \int_{a}^{x} \frac{|f^{(n+1)}(t)|}{n!} |x-t|^n \, dt \leq \int_{a}^{x} \frac{M}{n!} |x-t|^n \, dt = \frac{M}{(n+1)!} |x-a|^{n+1}.
\]

**Corollary:** Writing \( M := \sup_{x \in N(a)} |f^{(n+1)}(x)| \), the error function

\[
|E_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1} \text{ on } N(a).
\]

In particular,

\[
\lim_{x \to a} \frac{E_n(x)}{(x-a)^n} = 0.
\]

**Definition:** If \( g(x) \to 0 \) for \( x \in N^*(a) \) and \( \lim_{x \to a} \frac{f(x)}{g(x)} = 0 \), then we write

\[
f(x) = o(g(x)).
\]

So in the corollary we have

\[
E_n(x) = o((x-a)^n).
\]
Why might we care about bounding the error?

1) So we can decide how closely \( T_n(x) \) approximates \( f(x) \) at particular values of \( x \). For instance, with \( f(x) = e^x \) at \( a = 0 \),
\[
|e - \sum_{k=0}^{n} \frac{1}{k!}| = |E_n(1)| \leq \frac{\sup_{t \in [0,1]} e^t}{(n+1)!} (1-0)^{n+1} = \frac{e}{(n+1)!} < \frac{3}{(n+1)!}.
\]

[To see that \( e < 3 \), notice that \( \sum_{k=0}^{n} \frac{1}{k!} = 1 + 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{n!} < (1 + 1 + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2^n}) - (\frac{1}{2} - \frac{1}{2}) = (3 - \frac{1}{2^n}) - \frac{1}{2^n} \), and
\[
e = \sum_{k=0}^{n} \frac{1}{k!} + E_n(1) < 3 - \frac{1}{2^n} - \frac{1}{12} + \frac{1}{n}! < 3 \text{ by choosing } n \text{ large enough that } \frac{1}{n!} < \frac{1}{12} \text{ — which would happen regardless of how big } e \text{ was. Alternatively, you could use step functions to show that } \int_{1}^{3} \frac{dx}{x} > 1.]

This allows us to show that, for example, \( \sum_{k=0}^{10} \frac{1}{k!} \) gives at least 6 decimal places of accuracy, since \( \frac{3}{11!} \approx 0.00000008 \).

2) So we can show that certain numbers are irrational.

Theorem 2: Given sequences \( a_n, b_n \in \mathbb{Z} \), and \( r \in \mathbb{R} \), suppose that for each \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) s.t.
\[
n \geq N \Rightarrow 0 < |r - \frac{b_n}{a_n}| < \frac{\varepsilon}{a_n}.
\]
Then \( r \notin \mathbb{Q} \).

Proof: If \( r = \frac{p}{q} \) (where \( p, q \in \mathbb{Z} \)), then taking \( \varepsilon = \frac{1}{q} \) gives
\[
0 < |\frac{p}{q} - \frac{b_n}{a_n}| < \frac{1}{q a_n} \Rightarrow 0 < |\frac{p a_n - b_n q}{q a_n}| < 1
\]

\( \Box \)
\[ r = e, \quad a_n = n!, \quad b_n = \sum_{k=0}^{n} \frac{n!}{k!}. \] Put \( f(x) = e^x \).

Of course \( b_n/a_n = \sum_{k=0}^{n} \frac{1}{k!} = T_n(1) \). So the bound on \( E_n(1) \) above gives
\[
|e - \frac{b_n}{a_n}| = |E_n(1)| < \frac{3}{(n+1)!} = \frac{3/(n+1)}{n!} = \frac{3/(n+1)}{a_n}.
\]
Taking \( n \geq N > \frac{3}{\varepsilon} \) makes \( \frac{3}{n+1} < \varepsilon \). Moreover, we know \( E_n(1) = \int_0^1 \frac{f^n(x)}{n!} \, dx = \frac{1}{n!} \int_0^1 e^x (1-x)^n \, dx > 0 \).

So the hypotheses of Theorem 2 hold, and thus \( e \in \mathbb{Q} \). \hfill \square

(3) In order to compute indeterminate forms (i.e., limits of type \( 0/0, \ 1^\infty, \ \text{etc.} \)).

We shall make use of the following rules for computing with little-o notation: \( o(g(x)) + o(g(x)) = o(g(x)) \); \( o(c g(x)) = o(g(x)) \) if \( c \neq 0 \); \( o(g(x) - o(g(x)) = o(f(x) - g(x)) \); \( o(a g(x)) = o(g(x)) \), and if \( \lim_{x \to a} g(x) = 0 \), \( \frac{1}{1+g(x)} = 1 - g(x) + g(x) \frac{g(x)}{1+g(x)} = 1 - g(x) + o(g(x)) \).

**Example** Compute \( L = \lim_{x \to 0} \frac{a^x - \beta^x}{x} \). [In the examples, \( a = 0 \).]

By the Corollary, taking \( f(x) = e^x \) we have \( E_1(t) = o(t) \).

So \( e^t = 1 + t + o(t) \), and substitution yields
\[
a^x - \beta^x = e^{x \log a} - e^{x \log \beta} = \left( 1 + x \log a + o(x) \right) - \left( 1 + x \log \beta + o(x) \right) = \left( \log \frac{a}{\beta} \right) x + o(x).
\]

\[ \Rightarrow \quad \frac{a^x - \beta^x}{x} = \log \left( \frac{a}{\beta} \right) + o(1) \Rightarrow L = \log \left( \frac{a}{\beta} \right) \]
Ex/ Compute \( L = \lim_{x \to 0} \frac{\log(1+bx)}{x} \) and \( L' = \lim_{x \to 0} \left( \frac{1+bx}{x} \right)^{1/x}, \ b \in \mathbb{R}. \)

Taking \( f(x) = \log(1+bx) \) we have \( f'(x) = \frac{b}{1+bx} \Rightarrow (T, f)(x) = bx. \)

So \( \log(1+bx) = bx + o(x) \Rightarrow \frac{\log(1+bx)}{x} = b + o(1) \Rightarrow L = b. \)

Hence also \( L = \lim_{x \to 0} e^{\frac{1}{x} \log(1+bx)} = e^{\lim_{x \to 0} \frac{1}{x} \log(1+bx)} = e^b \) by continuity of \( e^p. \)

Ex/ Find \( (T, \tan)(x^1). \)

We have \( \sin(x) = x - \frac{x^3}{6} + o(x^4), \ \cos(x) = 1 - \frac{x^2}{2} + o(x^3) \)

\[ \Rightarrow \frac{\sin(x)}{\cos(x)} = \frac{1}{1 - \left( \frac{x^2}{2} + o(x^3) \right)} = \left(1 + \frac{x^2}{2} + o(x^2) \right) \]

\[ \Rightarrow \tan(x) = \frac{\sin(x)}{\cos(x)} = \left( x - \frac{x^3}{6} + o(x^4) \right) \left(1 + \frac{x^2}{2} + o(x^2) \right) \]

\[ = x + \frac{x^3}{3} + o(x^3). \]

The Taylor polynomials for \( \tan(x) \) are generally quite mysterious. For instance, \( T_7 \) is \( x + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \frac{272x^7}{7!}. \) The numerators appearing in the numerators are of course the derivatives of \( \tan \) at \( 0, \) the even ones being zero. The odd ones are given by a beautiful formula involving Bernoulli numbers:

\[ \tan^{-1}(0) = (-1)^{m-1} \left( \frac{4}{2m} - \frac{4^2m - 4m}{2m} \right) B_{2m}. \]