Lecture 27: 1st-order ODEs

An ordinary differential equation (ODE) is an equation of the form

$$\sum_{i=0}^{n} F_i(x,y) \frac{dy}{dx_i} = 0$$

from which one tries to determine $y = f(x)$ as a function of $x$.

The order of the equation is the highest derivative of $y$ occurring.

Today we'll discuss first-order ($n=1$) ODEs, i.e.

$$F_1(x,y) \frac{dy}{dx} + F_0(x,y) y = 0 \quad \iff \quad y' = F(x,y). \quad (\ast)$$

Ex 1: $y' = F(x) \quad \iff \quad y = \int_0^x F(u) \, du + C$.  //

Ex 2: $y' = ky$ with initial condition $f(0) = C$.

($f'(x) = kf(x)$)

know that $g(x) = Ce^{kx}$ is one solution; if $k$ is constant,

consider $h = \frac{f}{g} = \frac{f(x) e^{-kx}}{C} \Rightarrow h' = \frac{e^{-kx}}{C}(f'(x) - kf(x)) = 0$ \Rightarrow $h \equiv h(0) = \frac{f(0)}{g(0)} = \frac{C}{C} = 1 \Rightarrow f = g$.

This is called exponential growth or decay. Something convenient to write $t$ in lieu of $x$: e.g. if $k = -\lambda < 0$, $f(t) = Ce^{-\lambda t}$ might represent the remaining mass of some radioactive substance.

Its half-life $T$ is defined by $\frac{1}{2} = \frac{f(T)}{f(0)} = e^{-\lambda T} \Rightarrow T = \frac{\log{2}}{\lambda}$.
We shall call (1) linear if it takes the form
\[ y' + P(x) y = Q(x) \]  (**)

on some interval \( I \subset \mathbb{R} \) on which we seek solutions \( y = f(x) \).

Ex 3 / Homogeneous case: \( Q(x) = 0 \), i.e.
\[ y' + P(x) y = 0 \]  (***)

This is the case when linear combinations of solutions are solutions. One obvious solution is \( y = 0 \) on \( I \).

Assume next that \( y \neq 0 \) on \( I \). Then \(-P(x) = \frac{y'}{y} = \frac{d}{dx} \log |y|\). If \( x \in I \), then \( \log |y| = - \int_a^x P(u) \, du + C \) \( \Rightarrow \) \( y = k e^{-\int_a^x P(u) \, du} \) given the initial value \( f(a) = b \), we have \( y = b e^{-\int_a^x P(u) \, du} \).

In fact, these are the only solutions. Consider (for any solution) \( h(x) = f(x) e^{A(x)} \), where \( A(x) = \int_a^x P(u) \, du \). Then
\[ h'(x) = f'(x) e^{A(x)} + f(x) A'(x) e^{A(x)} = e^{A(x)} \left[ f' + Pf \right] = 0 \]
\( \Rightarrow \) \( h \equiv h(a) = f(a) e^{A(a)} = f(a) \Rightarrow f(x) = f(a) e^{-A(x)} \).

General (inhomogeneous or non-homogeneous) case, when \( Q(x) \neq 0 \):
let \( f \) be any solution of (1), \( A(x) = \int_a^x P(u) \, du \), and \( h(x) = f(x) e^{A(x)} \).
Then \( h' = (f' + Pf) e^{A} = Q e^{A} \iff h(a) = \int_a^x Q(u) e^{A(u)} \, du + k \)
\( \Rightarrow f = h e^{-A} = k e^{-A(a)} + e^{-A(x)} \int_a^x Q(u) e^{A(u)} \, du \), where \( k = f(a) \).

(1) uniquely determines the constant. This also highlights the fact that solutions to inhomogeneous equations all differ by solutions to the homogeneous equation (***).
Ex 4/ (Falling body with air resistance)

Let \( y = v(t) \) be the vertical velocity, \( s = \text{vertical position}, m = \text{mass}, a = y' = \text{acceleration}. \) We have \( ma = -mg - kv \) \((k, g > 0)\)

\[ \Rightarrow a = -g - \frac{k}{m} v \Rightarrow v' + \frac{k}{m} v = -g. \]

Writing

\[ A(t) := \int_0^t P(u) du = \int_0^t \frac{k}{m} v(u) du = \frac{kt}{m}, \]

\((t)\) gives

\[ v(t) = Ce^{-\frac{kt}{m}} - e^{-\frac{kt}{m}} \int_0^t g e^{-\frac{k(u-t)}{m}} du = Ce^{-\frac{kt}{m}} - \frac{gm}{k}, \]

so if \( v(0) = 0 \) then

\[ v(t) = \frac{gm}{k} \left( e^{-\frac{kt}{m}} - 1 \right) \Rightarrow \]

\[ s(t) = \int_0^t v(u) du + s(0) = s(0) + \frac{gm^2}{k^2} (1 - e^{-\frac{kt}{m}}) - \frac{gm}{k}. \]

and \( a(t) = v'(t) = -ge^{-\frac{kt}{m}}. \) Notice that \( a \to 0 \) and \( v \to -\frac{gm}{k} \) (limiting velocity) as \( t \to \infty \). This has a rather different character from the solution for \( k = 0 \) (no resistance), namely

\[ s(t) = s(0) - \frac{1}{2} gt^2. \]

Ex 5/ (Diluting a saltic solution)

Given: tank w/ 100 gallons brine \((k_0 \text{ lbs of salt/gallm})\)

5 gallons/min. of weaker brine \((k_i \text{ lbs/gallm})\) is flowing into tank

5 gallons/min. is flowing out of the tank

Determine amount of salt \( y = f(t) \).

Clearly \( f(0) = 100k_0 \). Inflow contributes \( 5k_i \text{ lbs/min.} \) to \( y \)

Outflow contributes \(-5 \left( \frac{y}{100} \right) \text{ lbs/min.} \) to \( y \).

\[ \Rightarrow y' + \frac{1}{20} y = 5k_i, \]

\[ y = Ce^{-\frac{t}{20}} + e^{-\frac{t}{20}} \int_0^t 5k_i e^{\frac{u}{20}} du = Ce^{-\frac{t}{20}} + 100k_i \]

\[ = 100k_0 e^{-\frac{t}{20}} + 100k_i (1 - e^{-\frac{t}{20}}). \]