Lecture 29: Complex numbers

A field is a set $F$ with:

- 2 commutative, associative binary operations $+$, $\cdot$.
- identity elements "0" for $+$, "1" for $\cdot$.
- inverses: $(-a)$ for $+$, $\left(\frac{1}{a}\right)$ for $\cdot$ if $a \neq 0$.
- distributive law.

$F$ is algebraically closed if for every polynomial equation

$$a_nz^n + a_{n-1}z^{n-1} + \cdots + a_2z + a_0 = 0$$

with each $a_i \in F$, there exists a solution (for $z$) in $F$.

Since $z^2 + 1 = 0$ has no real solution, the field $\mathbb{R}$ of real numbers is not algebraically closed.

Define the field of complex numbers by

$$\mathbb{C} := \{a+bi \mid a, b \in \mathbb{R}\} \quad (\cong \mathbb{R}^2 \text{ as a set})$$

with

- addition given by $(a_1+b_1i) + (a_2+b_2i) := (a_1+a_2) + (b_1+b_2)i$ (and additive inverse $-(a+bi) := (-a) + (-b)i$)
- multiplication given by $(a_1+b_1i) \cdot (a_2+b_2i) := (a_1a_2 - b_1b_2) + (a_1b_2 + a_2b_1)i$

For the mult. inverse, note that $(a+bi)(a-bi) = a^2 + b^2$, so that

$$(a+bi) \left(\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i\right) = 1.$$
The map $\mathbb{R} \to \mathbb{C}$ presents $\mathbb{R}$ as a subfield of $\mathbb{C}$, 
$v \mapsto v + 0i$ (which we just write “$v$”) much as $\mathbb{Q}$ is a subfield of $\mathbb{R}$. (This map respects $+$, $\cdot$.) We visualize $\mathbb{R} \subset \mathbb{C}$ via the picture 

![Graph showing the visualization of $\mathbb{R} \subset \mathbb{C}$]

which also shows that addition corresponds to adding ordered pairs or “vectors” (arrows) in $\mathbb{R}^2$. We can also define

- **complex conjugation**: flip about $\mathbb{R}$-axis  
  $a + bi := a - bi$  
  This respects $+$, $\cdot$: given $\alpha, \alpha_2 \in \mathbb{C}$,  
  $(\alpha_1 + \alpha_2) = \overline{\alpha_1 + \alpha_2}$ and $(\alpha_1 \cdot \alpha_2) = \overline{\alpha_1} \cdot \overline{\alpha_2}$ (check).  
  We also have $\overline{\overline{\alpha}} = \alpha$.

- **real & imaginary parts**: given $\alpha = a + bi \in \mathbb{C}$, set $\Re(\alpha) := a = \frac{1}{2}(\alpha + \overline{\alpha})$ (check)  
  $\Im(\alpha) := b = \frac{1}{2i}(\alpha - \overline{\alpha})$

- **modulus (or absolute value)**: $|\alpha| = \sqrt{a^2 + b^2}$ = distance from 0 to $\alpha$  
  Notice that $\alpha \overline{\alpha} = (a + bi)(a - bi) = a^2 + b^2 = (ab - ba)i = |\alpha|^2$  
  \[ \alpha \cdot \frac{\overline{\alpha}}{|\alpha|^2} = 1 \Rightarrow \alpha^{-1} = \frac{\overline{\alpha}}{|\alpha|^2} \]  
  Thus gives a simple way of thinking about (even visualizing) the matrix inverse. But in practice  
  To compute inverses & quotients we just write  
  \[
  \frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{ac + bd + (bc - ad)i}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i.
  \]
Properties of modulus:\n\[ |z| = |\bar{z}| \quad \text{is obvious;} \quad \text{and} \quad |z\bar{z}| = |z||\bar{z}|. \]

More interesting is the triangle inequality:\n\[ |z + \bar{z}| \leq |z| + |\bar{z}|. \]

[Proof: For \( z = x + iy \in \mathbb{C}, \ \text{Re}(z) = x \leq |z| = \sqrt{x^2 + y^2} = |z| \). Now \( |z + \bar{z}|^2 = (z + \bar{z})(\bar{z} + z) = \bar{z}z + \bar{z}z + \bar{z}z + \bar{z}z \), i.e. \( \bar{z}z \)
\[ = |z|^2 + |\bar{z}|^2 + 2 \text{Re}(z\bar{z}) \leq 2 |z||\bar{z}| = 2 |z|^2 |\bar{z}| = 2 |z||\bar{z}| \]
\[ \leq |z|^2 + |\bar{z}|^2 + 2 |z|^2 |\bar{z}| = (|z| + |\bar{z}|)^2. \quad \text{Take } \sqrt{\cdot} \].]

The argument of a complex number: \( \text{arg}(z) := \text{angle the segment from } 0 \text{ to } z \text{ makes with } x \)-axis.]}

Notice that if \( a > 0 \) then
\[ \text{arg}(a + bi) = \arctan \left( \frac{b}{a} \right). \]

If \( a < 0 \) then you have to add or subtract \( \pi \). I should also mention that \( \text{arg}(z) \) is only defined up to integer multiples of \( 2\pi \); that is, for our purposes here \( 2\pi \equiv 0 \).
(This is called working modulo \( 2\pi \) or \( 2\pi \mathbb{Z} \).)

Upshot: Writing \( \Theta := \text{arg}(z) \) and \( r = |z| \) then \( z = r \cos \Theta \) and \( b = r \sin \Theta \), hence \( z = r \left( \cos \Theta + i \sin \Theta \right) \). (Polar form, v. 1)

Here is the key property of arguments of complex numbers:

Proposition: If \( z_1, z_2 \) have arguments \( \Theta_1, \Theta_2 \) then \( z_1z_2 \) has argument \( \Theta_1 + \Theta_2 \mod 2\pi \). (and so \( z_1/z_2 \) has argument \( \Theta_1 - \Theta_2 \)).

Proof: Since \( |z_1z_2| = |z_1||z_2| \), this reduces to checking that
\[ (\cos \Theta_1 + i \sin \Theta_1) \cdot (\cos \Theta_2 + i \sin \Theta_2) = \cos(\Theta_1 + \Theta_2) + i \sin(\Theta_1 + \Theta_2). \]

But this is just the addition laws
\[ \begin{cases} 
\cos(\Theta_1 + \Theta_2) = \cos \Theta_1 \cos \Theta_2 - \sin \Theta_1 \sin \Theta_2 \\
\sin(\Theta_1 + \Theta_2) = \sin \Theta_1 \cos \Theta_2 + \cos \Theta_1 \sin \Theta_2 
\end{cases} \]
Ex/ Find $\sqrt{3+4i}$.

Let's think about square roots in general: if $d = r (\cos \theta + i \sin \theta)$, then the Proposition tells us that $(r^{\frac{1}{2}} (\cos \frac{\theta}{2} + i \sin \frac{\theta}{2}))^2 = \alpha$ hence that $\alpha^{\frac{1}{2}} = r^{\frac{1}{2}} \cos \frac{\theta}{2} + ir^{\frac{1}{2}} \sin \frac{\theta}{2} = r^{\frac{1}{2}} \sqrt{1 + \cos \theta} + i r^{\frac{1}{2}} \sqrt{1 - \cos \theta}

= \frac{\sqrt{r + r \cos \theta}}{2} + i \frac{\sqrt{r - r \cos \theta}}{2}

= \sqrt{\frac{r + r \cos \theta}{2}} + i \sqrt{\frac{r - r \cos \theta}{2}} = \sqrt{\frac{\sqrt{a^2 + b^2} + a}{2}} + i \sqrt{\frac{\sqrt{a^2 + b^2} - a}{2}}.

(Actually, $-\alpha^{\frac{1}{2}}$ will also be a square root too.) Applying this formula, $\sqrt{3+4i} = \sqrt{\frac{\sqrt{3^2 + 4^2} + 3}{2}} + i \sqrt{\frac{\sqrt{3^2 + 4^2} - 3}{2}} = \sqrt{\frac{\sqrt{25} + 3}{2}} + i \sqrt{\frac{\sqrt{25} - 3}{2}} = \sqrt{4} + i \sqrt{1} = 2 + i$.

Check: $(2+i)^2 = 2^2 + i^2 + 2i + 2i = 3 + 4i$.

You may be wondering still, since I mentioned that $\mathbb{R}$ was not algebraically closed, about $C$:

Fundamental Theorem of Algebra: $\mathbb{C}$ is algebraically closed.

We can't prove this right now, but I should point out that for a quadratic equation, the existence of complex solutions is clear:

$$A2^2 + B2 + C = 0 \quad (A, B, C \in \mathbb{C})$$

$$z = -\frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$

always exists in $\mathbb{C}$.

Now suppose $A, B, C \in \mathbb{R}$.
If one solution is non-real ($b/c \ B^2 - 4AC < 0$), then so is the other, and the two solutions are conjugate!

That is, $A2^2 + B2 + C = A (z - \bar{z})(z - \bar{z}) = A (z - \bar{z}) \bar{z} + \bar{z} z$

- So conversely, if roots are conjugate, the equation has real coefficients.