Lecture 3: Sets of real numbers

Set notation: Sets are collections of elements.

- "roster notation": \( A = \{2, 3, 4, 5, 6\} \) (list the elements)
  - write \( 2 \in A \) to say "2 is an element of A"
  - write \( \{2\} \subseteq A \) to say "\{2\} is a subset of A"
  - (other subsets of \( A \): \( \{2, 4\}, \{3, 5, 6\}, A \) itself, \( \emptyset \))

- "predicate notation": \( A = \{ x \in \mathbb{Z} \mid 2 \leq x \leq 6 \} \)
  - this requires \( A \) to be a subset of the "domain". Some common sets of numbers:
    - \( \mathbb{Z} \) (integers), \( \mathbb{Q} \) (rationals), \( \mathbb{R} \) (reals), \( \mathbb{C} \) (complex nos.)

- building sets from other sets: given sets \( A \) and \( B \),
  - \( A \cup B \) = union = \( \{ x \mid x \in A \text{ or } x \in B \} \)
  - \( A \cap B \) = intersection = \( \{ x \mid x \in A \text{ and } x \in B \} \)
  - \( A - B \) or \( A \setminus B \) = difference = \( \{ x \mid x \in A \text{ and } x \notin B \} \)

Can do the same for a collection (or "family") \( \mathcal{F} = \{ A_1, \ldots, A_n \} \):
  - \( \bigcup_{k=1}^{n} A_k \), \( \bigcap_{k=1}^{n} A_k \)

The real numbers

These are defined axiomatically in the text:

- Field axioms (commutativity, associativity, distributivity, negatives & inverses [except \( 1/0 \) !], etc.) — true for \( \mathbb{Q} \), \( \mathbb{R} \), \( \mathbb{C} \)
• **Order axioms:** existence of a subset of positive numbers closed under addition and multiplication, such that every \( x \) has \( x = 0, x > 0 \) or \( x < 0 \). (Define \( a < b \) if \( b - a \) is positive.) — true for \( \mathbb{Z}, \mathbb{Q}, \mathbb{R} \).

Here's a consequence — ** Transitive law:** if \( a < b \) and \( b < c \), then \( a < c \).

Proof: \( b - a > 0 \) and \( c - b > 0 \) \( \Rightarrow \) their sum is \( > 0 \), i.e., \((b - a) + (c - b) = c - a > 0 \).

• **The least-upper-bound axiom:** (this one is only true for \( \mathbb{R} \)) Suppose \( S \subset \mathbb{R} \) is nonempty & bounded above: i.e., \( \exists b \in \mathbb{R} \) such that \( s \leq b \) for every \( s \in S \). Then \( S \) has a least upper bound — that is, \( B \in \mathbb{R} \) such that (i) \( B \) is an upper bound for \( S \) (ii) no number less than \( B \) is an upper bound for \( S \).

We write \( \sup S := B \)

Similarly, if \( S \) is bounded below, then it has a greatest lower bound or *infimum* \( \inf S := -\sup(-S) \).

These axioms define \( \mathbb{R} \); one then defines

- the positive numbers \( \mathbb{P} \) as in lecture 1 (basically, \( 1, 2, 1, 1, 1, 1, \ldots \))
- the integers \( \mathbb{Z} = \mathbb{P} \cup \{ 0 \} \cup -\mathbb{P} \)
- the rationals \( \mathbb{Q} := \{ a / b \mid a, b \in \mathbb{Z} \text{ and } b \neq 0 \} \)

Of course, \( \mathbb{P} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \).

**Archimedean Property:** If \( x \in \mathbb{R}^+ \), \( y \in \mathbb{R} \), then \( \exists n \in \mathbb{P} \) s.t. \( nx > y \).

Proof: \( \mathbb{P} \) is not bounded above (otherwise, then \( \beta := \sup \mathbb{P} \) and \( m \in \mathbb{P} \) with \( m > \beta - 1 \) \( \Rightarrow \) \( m + 1 > \beta \)). So there must be an element of \( \mathbb{P} \) bigger than \( y / x \). \( \square \)
Ex. Let \( R := \{ x \in \mathbb{R} \mid x^2 < 2 \} \). This is bounded above (by, say, 2), and hence so has a least upper bound \( B := \text{sup} R \).

There are only 3 possibilities: \( B^2 < 2 \), \( B^2 > 2 \), or \( B^2 = 2 \). If we can rule out the first two, then the third holds, and \( B \) is a square root of 2.

- Suppose \( B^2 > 2 \). Let \( c := B - \frac{B^2 - 2}{2B} = \frac{B + 2}{2} \), so that \( 0 < c < B \) and

\[
    c^2 = B^2 - 2 + \frac{(B^2 - 2) \cdot 2}{4B^2} = 2 + \frac{(B^2 - 2)^2}{4B^2} > 2
\]

\( \Rightarrow \) \( c \) is an upper bound for \( R \)
\( \Rightarrow \) \( B \leq c \) \( \checkmark \)
(\( B \) least \( UB \))

- Suppose \( B^2 < 2 \). Let \( c \in \mathbb{R}^+ \) be less than \( B \) and \( \frac{2-B^2}{3B} \), so that \( (B+c)^2 = B^2 + 3Bc < B^2 + (2-B^2) = 2 \)

\( \Rightarrow \) \( B+c \in \mathbb{R} \)
\( \Rightarrow \) \( B+c \leq B \) \( \Rightarrow \) \( c \leq 0 \) \( \checkmark \)
(\( B \) \( UB \))

**Upshot:** \( \sqrt{2} \) exists in \( \mathbb{R} \). But not in \( \mathbb{Q} \):

---

Ex. Let \( S := \{ x \in \mathbb{Q} \mid x^2 < 2 \} \). I claim that \( S \) does not have a least upper bound. We need 2 facts:

- \( \sqrt{2} \) is irrational: Suppose there was a rational number \( \frac{a}{b} \in \mathbb{Q} \) with \( \left( \frac{a}{b} \right)^2 = 2 \). We may assume that \( a \) or \( b \) is odd, since otherwise we can cancel pairs of 2 until this is true (why?).

\( \Rightarrow a^2 = 2b^2 \Rightarrow a \) can't be odd \( \Rightarrow a \) even = \( 2c \), \( c \in \mathbb{P} \)

\( \Rightarrow 4c^2 = 2b^2 \Rightarrow 2c^2 = b^2 \Rightarrow b \) even. Contradiction.
If $p, r \in \mathbb{R}$ and $p < r$, then $\exists q \in \mathbb{Q}$ with $p < q < r$.

I'll prove this tomorrow: it's a consequence of the Archimedean property together with the "well-ordering principle".

So now, let $r \in \mathbb{Q}$ be a least upper bound for $S$, and put $p = \sqrt{2}$. We can't have $r = p$ (since $p \notin \mathbb{Q}$).

Suppose $r < p$; then $\exists t \in \mathbb{Q}$ between them, so that $t^2 < p^2 = 2 \Rightarrow t \in S$ but $r < t$, impossible since $r$ is an upper bound for $S$. So we are left with $r > p$; but then any $s \in \mathbb{Q}$ between them is an upper bound for $S$. Contradiction. //

Upshot: $\mathbb{Q}$ does not satisfy the least upper bound axiom.

(It isn't "close" enough.)

Properties of $\inf$ & $\sup$ (Apostol, pp. 26-28)

Lemma: If $a, x, y \in \mathbb{R}$ satisfy $a \leq x \leq a + \frac{y}{n}$ for all $n \in \mathbb{N}$, then $x = a$.

Proof: By the Archimedean property, if $x > a$ then $\exists n \in \mathbb{N}$ s.t. $n(x-a) > y$, i.e. $x > a + \frac{y}{n}$, clearly $x > a$ is false, so $x = a$. □

Property I: If $S \subset \mathbb{R}$ has $\text{UB}$, and $h \in \mathbb{R}^+$, then $\exists x \in S$ with $x > \text{sup} S \setminus \text{LB}$.

Proof: Otherwise $\text{sup} S - h$ is an $\text{UB}$ for $S$, impossible by $h \leq \text{sup} S$ is least.

Property II: Given $A, B \subset \mathbb{R}$ with $\text{UB}$, $C := \{atb : a \in A, b \in B\}$, we have $\sup C = \sup A + \sup B$. (same for $\text{inf}$)

Proof: Since $\sup A + \sup B$ is an $\text{UB}$ for $C$ (why?)
\[ \sup C \leq \sup A + \sup B. \] By property I, for any \( n \in \mathbb{N} \) exists \( a, b \in A, B \) such that \( a > \sup A - \frac{1}{n} \) and \( b > \sup B - \frac{1}{n} \). Then:

\[ \sup A + \sup B < a + b + \frac{2}{n} \leq \sup C + \frac{2}{n}. \]

Now apply the lemma! \( \square \)

**Property III:** If \( S, T \subset \mathbb{R} \) are nonempty and \( \forall s \in S, t \in T \) we have \( s < t \), then \( \sup S \leq \inf T \).

**Proof:** Every \( t \in T \) is an UB for \( S \), so \( \sup S \leq t \) (sets are finite!)

\[ \Rightarrow \sup S \text{ is a LB for } T \Rightarrow \sup S \leq \inf T. \]

(\( \inf T \) is greatest LB.) \( \square \)