Lecture 3: Sets of real numbers

Set notation: Sets are collections of elements.

- "roster notation": \( A = \{2, 3, 4, 5, 6\} \) (list the elements)
  
  write \( 2 \in A \) to say "2 is an element of A"

  write \( \{2\} \subseteq A \) to say "\{2\} is a subset of A"

  (other subsets of A: \( \{2, 4\} \), \( \{3, 5, 6\} \), A itself, \( \emptyset \))

- "predicate notation": \( A = \{ x \in \mathbb{Z} \mid 2 \leq x \leq 6 \} \)
  
  this requires \( A \) to be a subset of the "domain". Some common sets of numbers:
  
  \( \mathbb{Z} \) (integers), \( \mathbb{Q} \) (rationals), \( \mathbb{R} \) (reals), \( \mathbb{C} \) (complex num.)

- building sets from other sets: given sets \( A \) and \( B \),
  
  \( A \cup B = \text{union} = \{ x \mid x \in A \text{ or } x \in B \} \)
  
  \( A \cap B = \text{intersection} = \{ x \mid x \in A \text{ and } x \in B \} \)
  
  \( A - B \text{ or } A \setminus B = \text{difference} = \{ x \mid x \in A \text{ and } x \notin B \} \)

  Can do the same for a collection (or "family") \( \mathcal{A} = \{ A_1, \ldots, A_n \} \):
  
  \( \bigcup_{A \in \mathcal{A}} A = \bigcup_{k=1}^n A_k \), \( \bigcap_{A \in \mathcal{A}} A = \bigcap_{k=1}^n A_k \)

The real numbers

These are defined axiomatically in the text:

- Field axioms (commutativity, associativity, distributivity, negatives & inverses [except for \( \frac{1}{0} \) !], etc.) — true for \( \mathbb{Q}, \mathbb{R}, \mathbb{C} \)
• **Order axioms**: existence of a subset of positive numbers closed under addition and multiplication, such that every $x$ has $x = 0$, $x > 0$ or $x < 0$. (Define $a < b$ if $b - a$ is positive.) — true for $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$

Here's a consequence — **TRANSITIVE LAW**: if $a < b$ and $b < c$, then $a < c$.

Proof: $b - a > 0$ and $c - b > 0$ $\Rightarrow$ their sum is $> 0$,

i.e. $(b - a) + (c - b) = c - a > 0$. $\Box$

• **The least-upper-bound axiom**: (this one is only true for $\mathbb{R}$) Suppose $S \subseteq \mathbb{R}$ is nonempty & bounded above: i.e., $\exists b \in \mathbb{R}$ such that $s \leq b$ for every $s \in S$. Then $S$ has a least upper bound — that is, $B \in \mathbb{R}$ such that

(i) $B$ is an upper bound for $S$

(ii) no number less than $B$ is an upper bound for $S$.

We write $\sup S := B$

Similarly, if $S$ is bounded below, then it has a greatest lower bound or infimum $\inf S := - \sup (-S)$.

These axioms define $\mathbb{R}$; one then defines

- the positive numbers $\mathbb{P}$ as in lecture 1 (basically, $1, 1+1, 1+1+1, \ldots$)
- the integers $\mathbb{Z} := \mathbb{P} \cup \{0\} \cup \{-\mathbb{P}\}$
- the rationals $\mathbb{Q} := \{a/b \mid a, b \in \mathbb{Z} \text{ and } b \neq 0\}$

Of course, $\mathbb{P} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

**Archimedean Property**: If $m \in \mathbb{R}^+$, $y \in \mathbb{R}$, then $\exists n \in \mathbb{P}$ s.t. $ny > y$.

Proof: $\mathbb{P}$ is not bounded above (otherwise, there $\beta := \sup \mathbb{P}$ and $m \in \mathbb{P}$ with $m > \beta - 1 \Rightarrow m + 1 > \beta$ $\forall \beta$). So there must be an element of $\mathbb{P}$ bigger than $y/x$. $\Box$
Ex. Let \( R := \{ x \in \mathbb{R} \mid x^2 < 2 \} \). This is bounded above (by, say, 2), and (by the Compactness Theorem) so has a least upper bound \( B := \sup R \).

There are only 3 possibilities: \( B^2 < 2 \), \( B^2 > 2 \), or \( B^2 = 2 \). If we can rule out the first two, then the third holds, and \( B \) is a square root of 2.

- Suppose \( B^2 > 2 \). Let \( c := B - \frac{B^2 - 2}{2B} = \frac{B + \frac{2}{B}}{2} \),
so that \( 0 < c < B \) and
\[
c^2 = B^2 - (B^2 - 2) + \left(\frac{B^2 - 2}{4B^2}\right)^2 = 2 + \left(\frac{B^2 - 2}{4B^2}\right)^2 > 2
\]
\( \Rightarrow \) \( c \) is an upper bound for \( R \)
\( \Rightarrow \) \( B \leq c \) \( \star \)

(\( B \leq \text{UB} \))

- Suppose \( B^2 < 2 \). Let \( c \in \mathbb{R}^+ \) be less than \( B \) and \( \frac{2 - B^2}{3B} \),
so that \( (B+c)^2 < B^2 + 3Bc < B^2 + (2-B^2) = 2 \)
\( \Rightarrow B+c \in R \)
\( \Rightarrow B+c \leq B \) \( \Rightarrow c \leq 0 \) \( \star \)

(\( B \leq \text{UB} \))

**Upshot:** \( \sqrt{2} \) exists in \( \mathbb{R} \). But not in \( \mathbb{Q} \):

Ex. Let \( S := \{ x \in \mathbb{Q} \mid x^2 < 2 \} \). I claim that \( S \) does **not** have a least upper bound. We need 2 facts:

- **"\( \sqrt{2} \) is irrational":** suppose there was a rational number \( \frac{a}{b} \in \mathbb{Q} \) with \( (\frac{a}{b})^2 = 2 \). We may assume that \( a \) or \( b \) is odd, since otherwise we can cancel powers of 2 until this is true (why?).
Now \( a^2 = 2b^2 \Rightarrow a \) can't be odd \( \Rightarrow a \) even = \( 2c \), \( c \in \mathbb{P} \)
\( \Rightarrow 4c^2 = 2b^2 \Rightarrow 2c^2 = b^2 \Rightarrow b \) even. Contradiction.
• If $p, r \in \mathbb{R}$ and $p < r$, then $\exists q \in \mathbb{Q}$ with $p < q < r$.

I'll prove this tomorrow: it's a consequence of the Archimedean property together with the "well-ordering principle".

• So now, let $r \in \mathbb{Q}$ be a least upper bound for $S$, and put $p = \sqrt{2}$. We can't have $r = p$ (since $p \notin \mathbb{Q}$).

Suppose $r < p$; then $\exists t \in \mathbb{Q}$ between them, so that $t^2 = r^2 < 2 \Rightarrow t \notin S$ but $r < t$, impossible since $r$ is an upper bound for $S$. So we are left with $r > p$; but then any $s \in \mathbb{Q}$ between them is an upper bound for $S$. Contradiction. \[\]

Upshot: $\mathbb{Q}$ does not satisfy the least upper bound axiom.

(It isn't "dense" enough.)

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Properties of inf & sup (Apostol, pp. 26-28)

Lemma: If $a, x, y \in \mathbb{R}$ satisfy $a \leq x \leq a + \frac{y}{n}$ for all $n \in \mathbb{N}$, then $x = a$.

Proof: By the Archimedean property, if $x > a$ then $\exists n \in \mathbb{N}$ s.t. $n(x - a) > y$, i.e. $x > a + \frac{y}{n}$. Clearly $x < a$ is false, so $x = a$. \[\]

Property I: If $S \subseteq \mathbb{R}$ has $\text{UB}$, and $h \in \mathbb{R}^+$, then $\exists x \in S$ with $x > \sup S - h$.

Proof: Otherwise $\sup S - h$ is an $\text{UB}$ for $S$, impossible by $\text{c sup} S$ is least.

Property II: Given $A, B \subseteq \mathbb{R}$ with $\text{UB}$, $C := \{ab \mid a \in A, b \in B\}$, we have $\sup C = \sup A \times \sup B$. (same for $\inf L$).

Proof: Since $\sup A + \sup B$ is an $\text{UB}$ for $C$ (why?),
\[ \sup C \leq \sup A + \sup B. \] By property I, for any \( n \in \mathbb{N} \) there exists \( a \in A \) such that \( a > \sup A - \frac{1}{n} \) and \( b \in B \) such that \( b > \sup B - \frac{1}{n} \), then

\[ \sup A + \sup B < ab + \frac{2}{n} \leq \sup C + \frac{2}{n}. \] Now apply the lemma! 

Property III: If \( S, T \subset \mathbb{R} \) are nonempty and \( \forall s \in S, t \in T \) we have \( s < t \), then \( \sup S \leq \inf T \).

Proof: Every \( t \in T \) is an upper bound for \( S \), so \( \sup S \leq t \) (\( \forall t \in T \)). \( \Rightarrow \) \( \sup S \) is a lower bound for \( T \) \( \Rightarrow \) \( \inf S \leq \inf T \). (\( \inf T \) is greatest lower bound)