Lecture 30: Euler's Theorem

You may have noticed a strange relationship between the Taylor polynomials of \( \cos, \sin, \) and \( \exp. \) Recall that

\[
\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + O(x^8)
\]

\[
\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + O(x^8)
\]

while

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + O(x^8).
\]

If we substitute \( ix \) \((i = \sqrt{-1})\) in for \( x \) and assume the \( O(x^8) \) remains valid, then we get

\[
e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \frac{x^8}{8!} + O(x^8)
\]

\[
= \cos(x) + i \sin(x) + O(x^8).
\]

This at least suggests that, were we to be able to define complex-valued functions of a complex variable by power series, we could expect to get

\[
(*) \quad e^{ix} = \cos(x) + i \sin(x)
\]

as a theorem. This observation, due to Euler, is consistent with the fact that

\[
(**) \quad (\cos(x) + i \sin(x)) \cdot (\cos(y) + i \sin(y)) = \cos(x+y) + i \sin(x+y),
\]

proved in the last lecture as a consequence of angle-addition formulas.
If \( f(0) = e^x \), then 
\[ f(x) = e^{\alpha x} + iv(x), \]
where \( \alpha = a + ib \in \mathbb{C} \).

We can also define derivatives of integrals of complex functions of a real variable.

Moreover, the polar form of a complex number simplifies to
\[ e^x + 2e^x = e^x \cdot e^{2x} = (e^x)^{1+2} \]
which we use in an immediate consequence of this definition and (see). 
As an immediate consequence of this definition and (see), we have (we now know unique) solutions \( \cos(x) \) and \( \sin(x) \).

The definition, then we should have
\[ e^x = F(y) + iG(y) \]
and
\[ e^{-i} = F'(y) + iG'(y) \]

If we go to have \( i \) as a different way...
\[ f(x) = ae^{ax} \cos bx - be^{ax} \sin bx \]
\[ \quad + i \left[ ae^{ax} \sin bx + be^{ax} \cos bx \right] \]
\[ = ae^{ax} (\cos bx + i \sin bx) + ibe^{ax} (\cos bx + i \sin bx) \]
\[ = a e^{ax} (\cos bx + i \sin bx) = a e^{ax} e^{ibx} = a e^{ax}. \]

As a consequence, we get a much simpler description of solutions \( y = f(x) \) to
\[ y'' + ay' + by = 0. \]
Let \( \alpha \) be a root of \( r^2 + ar + b = 0 \); then writing \( g(x) = e^{\alpha x} \) we get
\[ g''(x) + ag'(x) + bg(x) = \alpha^2 e^{\alpha x} + \alpha a e^{\alpha x} + be^{\alpha x} \]
\[ = (\alpha^2 + \alpha a + b) e^{\alpha x} = 0. \]

If the characteristic equation factors as \((r-\alpha)(r-\bar{\alpha})\), then the general solution of \((***)\) becomes
\[ y = f(x) = Y_1 e^{\alpha x} + Y_2 e^{\bar{\alpha} x}, \quad \delta_1, \delta_2 \in \mathbb{C} \]
\[ = Y_1 e^{-\frac{\alpha}{2} x} e^{i\delta x} + Y_2 e^{-\frac{\alpha}{2} x} e^{-i\delta x} \]
\[ \alpha' = -\frac{\alpha}{2} + i\delta \]
\[ = e^{-\frac{\alpha}{2} x} \left( Y_1 e^{i\delta x} + Y_2 e^{-i\delta x} \right) \]
\[ = e^{-\frac{\alpha}{2} x} \left( (Y_1 + Y_2) \cos(\delta x) + i (Y_1 - Y_2) \sin(\delta x) \right). \]
Solving \( Y_1 + Y_2 = c_1 \), \( i(Y_1 - Y_2) = c_2 \) \((c_1, c_2 \in \mathbb{R})\) then recovers all real solutions to \((***)\).

\( \dagger \) If \( a, b \in \mathbb{R} \) then the other root is the conjugate of \( \alpha \); in fact,
\[ (r - \alpha)(r - \bar{\alpha}) = r^2 - 2Re(\alpha r + \alpha) + |\alpha|^2 = r^2 + ar + b \Rightarrow \alpha = -a/2 + i\delta, \]
where \( \delta = \frac{\sqrt{a^2 - 4b}}{2} \).
Powers of a complex number: writing $z = re^{i\theta}$ in polar form, we have $z^n = r^n(e^{i\theta})^n = r^n e^{in\theta}$. The last step here is by induction: $A(n)$ means $(e^{i\theta})^n = e^{in\theta}$. $A(1)$ is obvious, and $A(n-1) \Rightarrow A(n)$ by $(e^{i\theta})^n = e^{i\theta}(e^{i\theta})^{n-1} = e^{i\theta}e^{i(n-1)\theta} = e^{i\theta+n-1\theta} = e^{in\theta}$. Notice that this says $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$, which is called de Moivre's Theorem. (See the HW for more.)

Roots of unity are solutions to $z^n = 1$ ("unity" means 1). Then we n of these, $\omega = e^{2\pi ik/n}$ ($k = 0, 1, \ldots, n-1$).

Sometimes one writes $S_n = e^{2\pi i/n}$ so that the $n^{th}$ roots are $S_n^k$.

Ex/ Find the $n^{th}$ roots in Cartesien form ($a + bi$) for $n = 2, 3, 4, \ldots$ [Challenge: Can you do $n=5$?]

Ex/ Find $1 + S_n + S_n^2 + \ldots + S_n^{n-1}$. Interpret this geometrically.

Complex logarithm. Given $z = re^{i\theta}$, we may write $z = e^{\log r + i\theta} = e^{\log |z| + i\arg(z)}$.

For this reason, it is natural to define (so that $e^{\log(z)} = z$)

$$\log(z) := \log|z| + i\arg(z).$$

This gives $\log(-1) = \pi i$, $\log(i) = \frac{\pi i}{2}$, etc. and has the property that $\log(ab) = \log(a) + \log(b)$ "mod $2\pi i$".

That's all for now on complex numbers — more on ODEs Wed.