Lecture 31: More on 2nd-order linear ODEs

Let's begin with some examples of applications, arising from vibrations —

- Undamped case: \[ y'' + k^2 y = 0 \]
  - \[ y = A \sin kt + B \cos kt \]
  - \[ \Theta = \arctan \left( \frac{B}{A} \right) \]
  - \[ C = \sqrt{A^2 + B^2} \]
  - \[ y(t) = C \left( \cos \theta \sin kt + \sin \theta \cos kt \right) = C \sin \left( kt + \Theta \right) \]
  - Amplitude, angular frequency, phase angle

- Damped case: \[ y'' + 2cy' + k^2 y = 0 \]
  - Discriminant \( d = 4(c^2 - k^2) \)
  - \( c > 0 \) (critical damping): \[ y = e^{-ct} (A + Bt) \]
  - \( d > 0 \) (overcritical damping): \[ y = Ae^{-ct} e^{-\sqrt{c^2 - k^2} t} + Be^{-ct} e^{\sqrt{c^2 - k^2} t} \]
  - \( d < 0 \) (undercritical damping): \[ y = e^{-ct} (A \sin (\sqrt{c^2 - k^2} t) + B \cos (\sqrt{c^2 - k^2} t)) \]
  - Allows some oscillation

- Damped + driving force: \[ y'' + 2cy' + k^2 y = F(t) \]
  - We don't know how to solve these yet. One place they arise is in simple electrical circuits with an inductor, a resistor, a capacitor & a voltage source: then the current \( I(t) \) satisfies
    \[ LI' + RI + \frac{1}{C} \int I \, dt = V(t) \]
    \[ \Rightarrow LI'' + RI' + \frac{1}{C} I = V(t) \]

Example: A door on a hinge satisfies \[ \Theta'' = -2\Theta - 3\Theta' \]

Write \( \omega = \Theta' \) for angular velocity. For what initial values \( \Theta_0, \omega_0 \) does the door "slam", i.e., have \( \omega < 0 \) at \( \Theta = 0 \)?
The solution of the inhomogeneous equation (1) uses a beautiful trick involving the Wronskian. Recall that if the general solution of the homogenous equation $y'' + 2cy' + k^2y = 0$ is $y = c_1v_1(t) + c_2v_2(t)$, and $f(t)$ is one solution of (1), then the general solution of (1) is $y = f(t) + c_1v_1(t) + c_2v_2(t)$.

So how to find one solution $f(t)$?

Write $L := \frac{d}{dt} + 2c\frac{d}{dt} + k^2$ for the linear differential operator so that (1) becomes

$$L y_j = F,$$

and consider a function of the form $f(t) := u_1(t)v_1(t) + u_2(t)v_2(t)$.

Note that $f'(t) = (u_1'v_1 + u_2'v_2) + (u_1v_1' + u_2v_2')$

$f''(t) = (u_1'v_1'' + u_2'v_2'') + (u_1v_1'' + u_2v_2'') + u_1v_1'' + u_2v_2''$

$\Rightarrow Lf = f'' + 2cf' + f$

$= u_1(v_1'' + 2c v_1' + v_1) + u_2(v_2'' + 2c v_2' + v_2)$

$+ (u_1'v_1' + u_2'v_2') + (u_1v_1' + u_2v_2') + 2c(u_1v_1'' + u_2v_2'')$

$\Rightarrow$ get $Lf = F$ if

$$\begin{cases}
  u_1'v_1' + u_2'v_2' = 0 \\
  u_1v_1'' + u_2v_2'' = F
\end{cases}$$

Multiply 1st eqn. by $v_1'$

2nd by $v_1$

$\Rightarrow (u_1'v_1v_1' + u_2'v_2v_1') = v_1 F$

$u_1'(v_2v_1' - v_2v_1) = -v_1 F$

$u_2' = v_1 F / w$

Multiply 1st eqn. by $v_2$

$\Rightarrow (u_1'v_1v_2 + u_2'v_2v_2') = v_2 F$

$u_1'(v_1v_2' - v_1v_2) = -v_2 F$

$u_1' = -v_2 F / w$
where \( W = v_1v_2' - v_2v_1' \). We therefore have

\[
(4.9c) \quad f(t) = -v_1 \int \frac{v_2' F}{W} \, dt + v_2 \int \frac{v_1' F}{W} \, dt
\]

Solve (4.9c). (There are special cases where less effort is required to find \( f \), like if \( F \) is a polynomial or \( e^{rt} \) times a polynomial. See Problem 8.16 of Apostol.)

Ex. \( y'' + y' - 2y = e^t + e^{2t} \)

First note that the homogeneous equation has characteristic eqn
\[ 0 = r^2 + r - 2 = (r + 2)(r - 1) \Rightarrow v_1 = e^t, \quad v_2 = e^{-2t}. \]

Now \( W = v_1v_2' - v_2v_1' = 2e^t e^{-2t} - e^{-2t} e^t = -3e^{-t} \Rightarrow \)

\[
(4.9c) \quad f(t) = -e^t \int \frac{e^{-2t}(e^{2t} + e^{2t})}{-3e^{-t}} \, dt + e^{-2t} \int \frac{e^{-t}(e^{2t} + e^{2t})}{-3e^{-t}} \, dt
\]

\[ = \frac{1}{3} e^t \int (1 + e^t) \, dt - \frac{1}{3} e^{-2t} \int (e^{3t} + e^{4t}) \, dt = \frac{1}{3} e^t (t + e^t) - \frac{1}{3} e^{-2t} \left( \frac{1}{3} e^{3t} + \frac{1}{4} e^{4t} \right) = \left( \frac{4}{3} + \frac{1}{9} \right) e^t + \frac{1}{4} e^{2t}. \]