Lecture 32: Nonlinear first-order equations

Technique #1: Separable equations

These are equations of the form
\[ y' = Q(x) R(y) . \]

Claim: We can just write
\[ \frac{dy}{dx} = Q(x) R(y) \] and \[ \frac{dy}{R(y)} = Q(x) dx \] \[ \Rightarrow \int \frac{dy}{R(y)} = \int Q(x) dx \]
(up to constant).

Proof: This requires justification since we don’t prove things using “differentials.” But the justification is easy:
Using \( y = f(\alpha) \), we write the equation as \( f'(x) = Q(\alpha) R(f(\alpha)) \)
or \( \frac{f'(x)}{R(f(\alpha))} = Q(\alpha) \). Integrating both sides (w.r.t. \( x \)) gives
\[ \int Q(\alpha) dx = \int \frac{f'(x)}{R(f(\alpha))} dx \Rightarrow \int \frac{dy}{R(y)} = \int Q(x) dx \]

Ex/
\[ y' + 2xe^y = 0 \]
\[ \Rightarrow -\int e^{-y} dy = \int 2x dx \]
\[ \Rightarrow e^{-y} = x^2 + C \]
\[ \Rightarrow y = -\log(x^2+C) \]

Ex/
Find the time required for the water level to drop to 1 foot? to empty? (Let \( y = f(t) \) = (null))

Begin by observing that water exiting the orifice is really just “removed” from the top of the water level in the tank — i.e., a water droplet of mass \( m \) has “lost” mg units of potential energy, so upon exit must have
\[ \text{kinetic energy } \frac{1}{2}mv^2 = mg y \Leftrightarrow \frac{dV}{dt} = \frac{-5}{3} \text{ ft}^3 \text{ s}^{-1} \]
\[ \frac{-5}{3} = \frac{-\frac{\sqrt{2g}}{2} \sqrt{y}}{\frac{3}{2} \text{ ft}^2} \Rightarrow y = \frac{40}{312.5} \text{ ft}^2 \text{ ft} \]
\[ CG = \frac{32.8 \text{ ft}}{32.8 \text{ ft}} \]
At the same time, \( \frac{dV}{dc} = \frac{dV}{dy} \frac{dy}{dx} = 4y' \), since \( \frac{dV}{dy} = \text{cross-sectional area} \).

So we get the separable equation \( y' = -\frac{10}{3.12^2} \frac{v}{y} \) \( \Rightarrow \int \frac{dy}{y} = -\int \frac{10}{3.12^2} dc \)
\( \Rightarrow 2\sqrt{y} = -\frac{10}{3.12^2} t + C \Rightarrow y = \left( \frac{-5}{3.12^2} t + C \right)^2, \ C = \sqrt{2} \) \( \Rightarrow \) when \( y = 1 \), \( t = \frac{12 \cdot 3}{5} (\sqrt{5} - 1) \approx 35.8 \) sec
when \( y = 0 \), \( t = \frac{12 \cdot 3}{5} \sqrt{2} \approx 122.2 \) sec

This is assuming no friction.

The book introduces a discharge coefficient \( \approx 0.6 \), and the effort is to divide these answers by 0.6, giving 59.6 sec, resp. 203.6 sec.

Technique #2: Using homogeneity of the RHS

Suppose we want to solve
\[ y' = F(x,y) \]
where \( F(tx,ty) = F(x,y) \) for all \( t \neq 0 \). Then solutions are all "dilutions" of each other: if \( y = f(x) \) solves it, i.e. \( f'(x) = F(x,f(x)) \), then consider \( \frac{y}{t} = f(x) \) (i.e. \( y = tf(x) \)): we have
\[ y' = \frac{dy}{dx} tf(x) = f'(x) = F(x,tf(x)) = F(x,y) \]

To find one solution, the trick is to substitute \( v = \frac{y}{x} \) (i.e. \( y = vx \)), which gives
\[ y' = F(x,y) \]
\[ x'v + xv' = F(x,vx) \]
\[ v + xv' = F(1,v) \]

\[ v' = \frac{F(1,v) - v}{x} \]
and then using Technique #1
\[ \int \frac{dv}{F(1,v) - v} = \int \frac{dx}{x} \]

Example: Consider the two equations
\[ (A) \ y' = \frac{y^2 - x^2}{2xy} \quad \text{and} \quad (B) \ y' = \frac{2xy}{x^2 - y^2} \]
The graph of a solution to a DE is called an integral curve. Of course, at a point \((x_0,y_0)\) on such a curve, \( y' \) is
the slope. Since the product of \( \frac{y^2}{2xy} \cdot \frac{2x^2}{x^2-y^2} = -1 \), the integral curves of these equations passing through any \((x_0, y_0)\) are orthogonal \(\text{(perpendicular)}\).

For (A), \(y = vx\) yields \(v + v' = \frac{x^2v^2 - x^2}{2x^2v} = \frac{v^2 - 1}{2v}\)
\(\Rightarrow xv' = \frac{1-v^2}{2v} = \frac{1}{2} \frac{v^2+1}{v} \Rightarrow \int 2vdv = \int \frac{dx}{x} \Rightarrow \log x + C_0 = -\log(v^2+1) \Rightarrow C = (v^2+1)\)
\(\Rightarrow C = y^2 + x^2 \Rightarrow (x - \frac{C}{2})^2 + y^2 = \frac{C^2}{4} \). These are circles of radius \(\frac{C}{2}\) centered at \((\frac{C}{2}, 0)\) — i.e.,

centered on \(x\)-axis & passing through the origin.

For (B), \(y = vx\) yields \(v + v' = \frac{2x^2v}{x^2 - x^2v^2} = \frac{2v}{1-v^2}\)
\(\Rightarrow xv' = \frac{v^3}{1-v^2} \Rightarrow \int \frac{dx}{x} = \int 2vdv = \int \frac{(1-v^2)dv}{\sqrt{1+v^2}} = \int (\frac{1}{v} - \frac{2v}{1+v^2})dv = \log v - \log(1+v^2) = \log(\frac{v}{1+v^2}) \Rightarrow x = \frac{Cy}{1+y^2} = \frac{Cy}{1+y^2} \Rightarrow x^2 + y^2 - C = 0\) gives circles passing through the origin & centered on \(y\)-axis. So we have proved that these two families of circles are orthogonal wherever they meet!