Lecture 33: Limits of sequences

Definition: (i) A sequence is a function $a : \mathbb{P} \to \mathbb{R}$ (or $\mathbb{P} \cup \{0\}$, or $\mathbb{C}$).

(Its values $a(n)$ are usually denoted a_n, and the sequence as a whole is denoted $\{a_n\}$ instead of "a".)

(ii) We write $\lim_{n\to\infty} a_n = L$ if $\forall \epsilon > 0 \ \exists N \in \mathbb{P}$ s.t. $n \geq N \Rightarrow |a_n - L| < \epsilon$. Say $\{a_n\}$ converges, write $a_n \to L$. If there is no such L, $\{a_n\}$ diverges.

(iii) We write $\lim_{n\to\infty} a_n = \infty$ if $\forall M > 0 \ \exists N \in \mathbb{P}$ s.t. $n \geq N \Rightarrow a_n > M$. (Note: this is a special kind of divergence.)

Lemma: (a) If $a_n = f(n)$ for some function f on $\mathbb{R}^+$, and $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$.

(b) If $\lim a_n$ & $\lim b_n$ exist, then $\lim (Ca_n + Db_n) = C \lim a_n + D \lim b_n$.
$\lim a_n b_n = \lim a_n \cdot \lim b_n$
$\lim a_n / b_n = \frac{\lim a_n}{\lim b_n}$ provided $\lim b_n \neq 0$.

(c) If $\lim a_n = L$ & $f(x)$ is continuous at L, then $\lim f(a_n) = f(L)$.

(d) If $\lim a_n = L = \lim b_n$ and $a_n \leq c_n \leq b_n$ for $n \geq N$, then $\lim_{n\to\infty} c_n = L$.

Proofs for (b)–(d) are the same as in the function case, and (a) is true by definition.

Ex/ (a) $\Rightarrow \lim n^r = \begin{cases} 0 & r < 0 \\ \infty & r > 0 \end{cases}$, $\lim \alpha^n = \begin{cases} 0 & |\alpha| < 1 \\ \infty & |\alpha| > 1 \end{cases}$.

We can also make use of L'Hôpital: e.g. $\lim_{n\to\infty} \left(1 + \frac{\beta}{n}\right)^n = e^\beta$ (why?)

or $\lim \frac{\log(n)}{\sqrt{n}} = \lim_{x\to\infty} \frac{\log(x)}{\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/2\sqrt{x}} = \lim_{x\to\infty} 2x^{-1/2} = 0$. //

Remark: Rather than "using limits of functions" via (e), we can also give direct proofs of these facts: e.g. for $\lim \alpha^n = 0$ for $|\alpha|<1$, given $\epsilon>0$ take $N \geq \frac{\log \epsilon}{\log|\alpha|}$ (>0). Then $n \geq N \Rightarrow n\log|\alpha| < \log\epsilon$ $\Rightarrow |\alpha|^n < \epsilon$. //

Ex/ $\lim_{n\to\infty} \frac{\sqrt{e^{2n}-e^{-2n}}}{1+e^n} = \lim_{n\to\infty} \frac{\sqrt{1-e^{-4n}}}{e^{-n}+1} \overset{(b,c)}{=} \frac{\sqrt{1-\lim e^{-4n}}}{\lim e^{-n}+1} = \frac{\sqrt{1-0}}{0+1} = 1.$

$\lim_{n\to\infty} \frac{1}{1+n^3} = \lim \frac{n^{-3}}{n^{-3}+1} = \frac{\lim n^{-3}}{\lim n^{-3}+1} = \frac{0}{0+1} = 0$

$\lim_{n\to\infty} \cos\left(\frac{n\pi}{2}\right)$ doesn't exist (why?)

$\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$ by (d) (not L'Hôpital) //

Problem: Which of $(-1)^n$, $\left(-\frac{1}{2}\right)^n$, 2^n, $\frac{n}{\log n}$, $\frac{n!}{n^n}$ converge?

Some sequences are given by a formula (viz., "$a_n = f(n)$"), but others are given by a recursion.

Ex/ A particular breed of rabbits is immortal; each month, each pair produces a new pair which becomes productive after 2 months. Fibonacci just bought a newborn pair from the pet store. After quite a few months, at what rate will they be multiplying?

If f_n = # of pairs at beginning of month n, then the rate is $a_n = \frac{f_{n+1}}{f_n}$. We want $\lim_{n\to\infty} a_n =: L$. (Assume this exists.)
Notice that $f_{n+1} = f_n + f_{n-1}$ (why?), so dividing by f_n

$\frac{f_{n+1}}{f_n} = 1 + \frac{f_{n-1}}{f_n} \Rightarrow a_n = 1 + \frac{1}{a_{n-1}} \Rightarrow \lim_{n\to\infty} a_n = 1 + \frac{1}{\lim_{n\to\infty} a_n}$

$\Rightarrow L = 1 + L^{-1} \Rightarrow L^2 - L - 1 = 0 \Rightarrow L = \frac{1+\sqrt{5}}{2}$,

the golden ratio! //

To actually prove limits of some types exist (like those in the problem) exist, we need a supplement to the Lemma.

Definition: Let $\{a_n\}$ be a sequence of real #s. Then we say that $\{a_n\}$ is monotonic if it is increasing ($a_{n+1} \geq a_n$) "↗" or decreasing ($a_{n+1} \leq a_n$) "↘" for $n \geq N$. (Note that you can test sequences of positive real #s for monotonicity by showing $\frac{a_{n+1}}{a_n} > 1$ or $a_{n+1} - a_n \gtrless 0$.)

Monotonic Sequence Theorem: If $\{a_n\}$ is increasing & bounded above, or decreasing & bounded below, then $\lim a_n$ exists.

(e.g. by zero; only need this for $n \geq N$)

Proof: Write (in the increasing case) $L := \sup\{a_n\}$. This exists since $\{a_n\}$ is bounded above. Let $\epsilon > 0$ be given. Since L is the least upper bound, $L - \epsilon$ is NOT an upper bound — i.e. there exists $N \in \mathbb{P}$ s.t. $a_N > L - \epsilon$.

Since $a_n \nearrow$, $n \geq N \Rightarrow L \underset{UB}{\geq} a_n \geq a_N > L - \epsilon \Longrightarrow$

$0 \geq a_n - L > -\epsilon \Rightarrow |a_n - L| < \epsilon$. Therefore $\lim a_n = L$. □

PROBLEM: try applying this to $\frac{n!}{n^n}$, or to $\left(\frac{1}{5}\right)^n \binom{2n}{n}$.

Ex/ Define a sequence by the recurrence relation $a_{n+1} = f(a_n)$, where $f(x) = \frac{1}{3-x}$, and $a_1 = 2$. Then $a_2 = 1$, $a_3 = \frac{1}{2}$, $a_4 = \frac{2}{5}$, ... appears to be decreasing and bounded below by 0.

[**Proof:** Inductively assume $0 < a_n < a_{n-1} < 3$: then

$3 > 3 - a_n > 3 - a_{n-1} > 0 \Rightarrow \frac{1}{3} < \underbrace{\frac{1}{3-a_n}}_{a_{n+1}} < \underbrace{\frac{1}{3-a_{n-1}}}_{a_n} \Longrightarrow$

$0 < a_{n+1} < a_n < 3$.

So this is true for all n.]

Applying the Theorem, $L := \lim a_n$ exists. It seems reasonable to expect that L should be a "fixed point" of f, i.e. $f(L_0) = L_0$

$\Rightarrow L_0 = \frac{1}{3-L_0} \Rightarrow L_0^2 - 3L_0 + 1 = 0 \Rightarrow L_0 = \frac{3 \pm \sqrt{9-4}}{2}$. The "+" solution is too big since the limit is clearly < 1. So let's try to prove that $L = L_0 := \frac{3-\sqrt{5}}{2}$. We consider

$$|a_{n+1} - L_0| = \left|\frac{1}{3-a_n} - \frac{1}{3-L_0}\right| = \frac{|a_n - L_0|}{|3-a_n|\,|3-L_0|}$$

WLOG: $a_n < 2 \Rightarrow 3 - a_n > 1$ and $3 - L_0 = \frac{3}{2} + \frac{\sqrt{5}}{2} > 2$ $\Big\} < \frac{1}{2}|a_n - L_0| \Rightarrow |a_{n+1} - L_0| < \frac{1}{2^n}|a_1 - L_0| \to 0$

Hence $\lim a_n = \frac{3-\sqrt{5}}{2}$. //

Definition: A series is a sequence $\{s_n\}$ given by the n^{th} partial sums of another sequence: $s_n = \sum_{k=1}^{n} a_k$.
If it converges to a limit S (i.e. $\lim_{n\to\infty} s_n = S$), we say the series converges with sum S, and write $\sum_{k=1}^{\infty} a_k = S$.
or just "$\sum$"

Ex/ The lemma above yields the property

- $\sum (Ca_k + Db_k) = C\sum a_k + D\sum b_k$ provided $\sum a_k$ & $\sum b_k$ exist

hence

- $\sum a_k$ converges + $\sum b_k$ diverges $\Rightarrow$ $\sum (a_k + b_k)$ diverges

(otherwise $\sum(a_k+b_k) + \sum(-a_k) = \sum b_k$ would converge) //

Ex/ $\sum \frac{1}{k}$ diverges: $s_n = \sum_{k=1}^{n} \frac{1}{k} \geq \int_1^{1+n} \frac{1}{x}\,dx = \log(1+n) \longrightarrow \infty$. //

Ex/ $\sum \alpha^k$ converges if $|\alpha| < 1$ (here α can be a complex #):

$$s_n = \sum_{k=0}^{n} \alpha^k \Rightarrow s_n - \alpha s_n = (1 + \alpha + \cdots + \alpha^n) - (\alpha + \cdots + \alpha^n + \alpha^{n+1})$$
$$= 1 - \alpha^{n+1}$$
$$\Rightarrow s_n = \frac{1-\alpha^{n+1}}{1-\alpha} = \frac{1}{1-\alpha} - \frac{\alpha^{n+1}}{1-\alpha} \longrightarrow \frac{1}{1-\alpha}.$$ //

We'll talk more about convergence & divergence of series in the remainder of this week.