Lecture 33: Limits of sequences

Definition: (i) A sequence is a function \( a : \mathbb{N} \rightarrow \mathbb{R} \) (Its values \( a(n) \) are usually denoted \( a_n \), and the sequence as a whole is denoted \( \{a_n\} \) instead of \( \{a\} \)).

(ii) We write \( \lim_{n \to \infty} a_n = L \) if \( \forall \varepsilon > 0 \exists N \in \mathbb{N} \) s.t. \( n \geq N \Rightarrow |a_n - L| < \varepsilon \). Say \( \{a_n\} \) converges, write \( a_n \to L \). If there is no such \( L \), \( \{a_n\} \) diverges.

(iii) We write \( \lim_{n \to \infty} a_n = \infty \) if \( \forall M > 0 \exists N \in \mathbb{N} \) s.t. \( n \geq N \Rightarrow a_n > M \). (Note: this is a special kind of divergence.)

Lemma: If \( a_n = f(n) \) for some function \( f \) on \( \mathbb{R}^+ \), and \( \lim_{x \to \infty} f(x) = L \), then \( \lim_{n \to \infty} a_n = L \).

(b) If \( \lim a_n \) and \( \lim b_n \) exist, then \( \lim (Ca_n + Db_n) = C\lim a_n + D\lim b_n \).

(c) If \( \lim a_n = L \) and \( f(x) \) is continuous at \( L \), then \( \lim f(a_n) = f(L) \).

(d) If \( \lim a_n = L = \lim b_n \) and \( a_n \leq c_n \leq b_n \) for \( n \geq N \), then \( \lim a_n = \lim b_n = L \).

Proofs for (b)-(d) are the same as in the function case, and (e) is true by definition.

Ex/Ex. \( \lim n^r = \begin{cases} 0 & r < 0 \\ \infty & r > 0 \end{cases} \) \( \lim \frac{a^n}{a} = \begin{cases} 0 & |a| < 1 \\ \infty & |a| > 1 \end{cases} \).

We can also make use of L'Hôpital: e.g. \( \lim_{n \to \infty} (1 + \frac{\theta}{n})^n = e^\theta \) (why?) or \( \lim \frac{\log(n)}{n^{1/2}} = \lim \frac{\log(n)}{n^{1/2}} = \lim \frac{\frac{1}{n}}{\frac{1}{2n^{1/2}}} = \lim 2x^{-1/2} = 0 \).
Remark: Rather than "using limits of functions" via (e), we can also give direct proofs of these facts: e.g. for \( \lim a^n = 0 \) for \( k < 1 \), given \( \varepsilon > 0 \) take \( N \geq \frac{\log \varepsilon}{\log k} (> 0) \). Then \( n \geq N \Rightarrow n \log k^x < \log \varepsilon \Rightarrow |a|^n < \varepsilon \).

Ex /
\[
\lim_{n \to \infty} \frac{\sqrt[n]{e^{2n}} - e^n}{1 + e^n} = \lim_{n \to \infty} \frac{\sqrt[1]{1 - e^{-2n}}}{e^{-n} + 1} = \frac{\sqrt[1]{1 - e^{-2n}}}{e^{-n} + 1} = \frac{\sqrt[1]{1 - e^{-2n}}}{e^{-n} + 1} = 1.
\]
\[
\lim_{n \to \infty} \frac{1}{1 + n^2} = \lim_{n \to \infty} \frac{n^3}{n^3 + 1} = \lim_{n \to \infty} \frac{n^3}{n^3 + 1} = \frac{0}{0+1} = 0
\]
\[
\lim_{n \to \infty} \cos \left( \frac{2n}{n} \right) \text{ doesn't exist (why?)}
\]
\[
\lim_{n \to \infty} \frac{\sin(n)}{n} = 0 \text{ by (d1) } \left( \text{not L'Hopital} \right)
\]

Problem: Which of \((-1)^n\), \((-\frac{1}{2})^n\), \(2^n\), \(\frac{n}{\log n}\), \(\frac{n!}{n^n}\) converge?

Some sequences are given by a formula (v.e., "\( a_n = f(n) \)"), but others are given by a recursion.

Ex / A particular breed of rabbits is immortal: each month, each pair produces a new pair which becomes productive after 2 months. Fibonacci just bought a newborn pair from the pet store. After quite a few months, at what rate will they be multiplying?

If \( f_n \) = # of pairs at beginning of month \( n \), then the rate is \( a_n = \frac{f_{n+2}}{f_n} \). We want \( \lim_{n \to \infty} a_n = L \). (Assume this exists.) Notice that \( f_{n+1} = f_n + f_{n-1} \) (why?), so dividing by \( f_n \),
\[
\frac{f_{n+1}}{f_n} = 1 + \frac{f_{n-1}}{f_n} \Rightarrow a_n = 1 + \frac{1}{a_{n-1}} \Rightarrow \lim_{n \to \infty} a_n = 1 + \frac{1}{\lim_{n \to \infty} a_n} \Rightarrow L = 1 + L^{-1} \Rightarrow L^2 - L - 1 = 0 \Rightarrow L = \frac{1 + \sqrt{5}}{2},
\]
the golden ratio!
To actually prove limits of some types exist (like those in the problem) exist, we need a supplement to the lemma.

**Definition:** Let \( \{a_n\} \) be a sequence of real numbers. Then we say that \( \{a_n\} \) is **monotonic** if it is increasing \( (a_{n+1} \geq a_n) \) or decreasing \( (a_{n+1} \leq a_n) \) for \( n \geq N \). (Note that you can test sequences of positive real numbers for monotonicity by showing \( \frac{a_{n+1}}{a_n} \))

\[ \leq 1 \text{ or } a_{n+1} - a_n \geq 0. \]

**Monotonic Sequence Theorem:** If \( \{a_n\} \) is increasing & bounded above, or decreasing & bounded below, then \( \lim_{n \to \infty} a_n \) exists.

**Proof:** Write (in the increasing case) \( L = \sup \{a_n\} \). This exists since \( \{a_n\} \) is bounded above. Let \( \varepsilon > 0 \) be given. Since \( L \) is the least upper bound, \( L - \varepsilon \) is not an upper bound — i.e., there exists \( N \in \mathbb{N} \) s.t. \( a_N > L - \varepsilon \).

Since \( a_n \uparrow \), \( n \geq N \Rightarrow L \geq a_n \geq a_N > L - \varepsilon \)

\[ 0 > a_n - L \leq - \varepsilon \Rightarrow |a_n - L| < \varepsilon. \]

Therefore \( \lim_{n \to \infty} a_n = L. \quad \square \)

**Problem:** Try applying this to \( \frac{n!}{n^n} \), or to \( \left( \frac{1}{5} \right)^n \binom{2n}{n} \).

**Ex:** Define a sequence by the recurrence relation \( a_{n+1} = f(a_n) \)

where \( f(x) = \frac{1}{2 - x} \), and \( a_1 = 2 \). Then \( a_2 = 1, \ a_3 = \frac{1}{2}, \ a_4 = \frac{2}{5}, \ldots \) appears to be decreasing and bounded below by 0.

**Proof:** Inductively assume \( 0 < a_n < a_{n-1} < 3 \) then

\[ 3 > 3 - a_n > 3 - a_{n-1} > 0 \Rightarrow \frac{1}{3} < \frac{1}{3 - a_n} < \frac{1}{3 - a_{n-1}} \Rightarrow \]

\[ 0 < a_{n+1} < a_n < 3. \]

So this is true for all \( n \).

**Applying the Theorem:** \( L = \lim_{n \to \infty} a_n \) exists. It seems reasonable to expect that \( L \) should be a "fixed point" of \( f \), i.e., \( f(L) = L \).
\[ L_0 = \frac{1}{3 - L_0} \Rightarrow L_0^2 - 3L_0 + 1 = 0 \Rightarrow L_0 = \frac{3 \pm \sqrt{9 - 4}}{2} \]. The "+" solution is too big since the limit is clearly < 1. So let's try to prove that \( L = L_0 := \frac{3 - \sqrt{5}}{2} \). We consider
\[
|a_n - L_0| = \left| \frac{1}{3 - a_n} - \frac{1}{3 - L_0} \right| = \frac{|a_n - L_0|}{|13 - a_n| |13 - L_0|}
\]
(we: \( a_n < 2 \Rightarrow 3 - a_n > 1 \) and \( 3 - L_0 = \frac{5}{2} > 2 \)) \( < \frac{1}{2} |a_n - L_0| \Rightarrow |a_{n+1} - L_0| < \frac{1}{2} |a_n - L_0| \to 0 \)

Hence \( \lim a_n = \frac{3 - \sqrt{5}}{2} \).

**Definition:** A series is a sequence \( \{s_n\} \) given by the \( n \)th partial sums of another sequence: \( s_n = \sum_{k=1}^{n} a_k \).

If it converges to a limit \( S \) (i.e., \( \lim s_n = S \)), we say the series converges with sum \( S \), and write \( \sum_{k=1}^{\infty} a_k = S \) or just \( \sum a_k = S \).

**Ex/** The lemma above yields the property
- \( \sum (C a_k + D b_k) = C \sum a_k + D \sum b_k \) provided \( \sum a_k \) and \( \sum b_k \) exist.
- \( \sum a_k \) converges + \( \sum b_k \) diverges \( \Rightarrow \sum (a_k + b_k) \) diverges (otherwise \( \sum (a_k + b_k) + \sum (-a_k) = \sum b_k \) would converge).

**Ex/** \( \sum \frac{1}{k} \) diverges: \( s_n = \sum_{k=1}^{n} \frac{1}{k} \geq \int_{1}^{n+1} \frac{1}{x} dx = \log (1 + n) \to \infty \).

**Ex/** \( \sum a_k \) converges if \( |a| < 1 \) (where \( a \) can be a complex number):

\[
\begin{align*}
S_n &= \sum_{k=0}^{n} a^k = (1 + a + \ldots + a^n) - (1 + a + \ldots + a^{n+1}) \\
&= 1 - a^{n+1} \\
\Rightarrow \quad S_n &= \frac{1 - a^{n+1}}{1 - a} = \frac{1}{1 - a} - \frac{a^{n+1}}{1 - a} \to \frac{1}{1 - a}.
\end{align*}
\]

We'll talk more about convergence & divergence of series in the remainder of this week.