Lecture 35: More convergence tests

Before we move on from the integral test, here is an interesting example of improper integrals.

\[ \text{Ex/ } \Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt \quad (s > 0) \] is called the Gamma function.

The integral converges: if \( s > 1 \), by comparison with \( \lim_{a \to \infty} \int_a^1 \frac{t^s}{a^s} \in \frac{1}{a} \), we have:

\[ \int_1^\infty \frac{t^s}{a^s} \in \frac{1}{a} \]

The integral is eventually smaller than \( \int_1^\infty t^{-2} \), and \( \int_1^\infty t^{-2}dt \) converges.

It has the really cool property that

\[ \Gamma(s+1) = \int_0^\infty t^s e^{-t}dt = -t^s \left. \frac{d}{dt} \right|_{t=0} + \int_0^\infty t^{s-1}e^{-t}dt = 0 + s \Gamma(s) \]

\[ \Gamma(s+1) = s \Gamma(s) \]

Since \( \Gamma(1) = \int_0^\infty e^{-t}dt = -e^{-t} \big|_0^\infty = 1 (\ne 0) \), \( \Gamma(n+1) = n! \).

\[ \Gamma(s) = \int_0^\infty e^{-t}dt = -e^{-t} \big|_0^\infty = 1 (\ne 0) \], \( \Gamma(n+1) = n! \).

(2) Ratio test: If \( a_n > 0 \) and \( \rho = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \),

- \( 0 < \rho < 1 \) \( \Rightarrow \) \( \sum a_n \) converges
- \( \rho = 1 \) : inconclusive (e.g., \( a_n = \frac{1}{n} \) vs. \( \frac{1}{n^2} \))
- \( 0 < \rho \) \( \Rightarrow \) \( \sum a_n \) diverges.

\[ \text{Ex/ } \sum \frac{2^n}{n!} \text{ has } \rho = \lim_{n \to \infty} \frac{2^{n+1}(n+1)!}{2^n/n!} = \lim_{n \to \infty} \frac{2}{n+1} = 0 \Rightarrow \sum \]
\[ \sum_{n=1}^{\infty} \frac{n!}{n^n} \quad \text{has} \quad p = \lim_{n \to \infty} \frac{(n+1)!}{(n+1)^n} \cdot \frac{n^n}{n!} = \lim_{n \to \infty} \frac{n^n}{(n+1)^n} = \frac{1}{\lim \left(1 + \frac{1}{n}\right)^n} = \frac{1}{e} \Rightarrow C \]

Proof: If \( p < 1 \), then taking \( r \in (p, 1) \) the exists \( N \) s.t.

\[ n \geq N \Rightarrow \frac{a_n}{a_n} < r = q_{n+1} < q_{n+1} < \ldots < r^k a_N \]

\[ \Rightarrow a_n < \frac{a_n}{r^n} = C \cdot r^n \Rightarrow \sum a_n \text{ converges by comparison to (convergent) geometric series } \sum a_n. \]

If \( p > 1 \), then for \( n \geq N \), \( a_n \) is increasing \( \Rightarrow a_n \) doesn't limit to 0 \( \Rightarrow \sum a_n \) diverges by basic divergence test. \( \square \)

9 Root test: If \( a_n > 0 \) and \( r = \lim_{n \to \infty} \sqrt[n]{a_n} \),

- \( 0 \leq r < 1 \) \( \Rightarrow \sum a_n \) converges
- \( r = 1 \) : inconclusive
- \( r > 1 \) (or \( \infty \)) \( \Rightarrow \sum a_n \) diverges.

Ex/ \[ \sum_{n=1}^{\infty} \frac{1}{(\log n)^n} \quad \text{has} \quad r = \lim_{n \to \infty} \sqrt[n]{\frac{1}{(\log n)^n}} = 0 \Rightarrow C \]

\[ \sum_{n=1}^{\infty} \frac{n!}{n^n} \quad \text{has} \quad a = \lim_{n \to \infty} \frac{n^n}{n!} = \frac{1}{e} \Rightarrow C \]

\[ \sum_{k=1}^{\infty} \log(k) \leq \int_{k=2}^{n} \log(x) \, dx \leq \sum_{k=2}^{n} \log(k) \quad (\text{draw the picture!}) \]

\[ \exp \left( \frac{\log((n-1)!!)}{(n-1)!} \right) \leq n \log n - n \leq \log(n!) \]

\[ \exp \left( \frac{\log((n-1)!!)}{(n-1)!} \right) \leq \frac{n^n}{e^n} \leq n! \leq \frac{n!}{e^n} \]

Proof: If \( a < 1 \), take \( r \in (a, 1) \). There exists \( N \) s.t.

\[ n > N \Rightarrow a_n < r \]

\[ \Rightarrow a_n < r^n \]. Since \( \sum a_n \) C, done by comp. test. The \( r > 1 \) case is as in the last proof. \( \square \)
Theorem: Alternating Series Test: A series of the form \( \sum_{n=1}^{\infty} (-1)^n b_n \) converges if \( b_n \) is decreasing with limit 0.

**Proof:**

\[
S_{2k} = S_{2k-2} + \frac{b_{2k}}{2} \quad \text{is increasing} \quad \Rightarrow \quad S_{2k} \geq \frac{b_{2k}}{2} > 0
\]

\[
S_{2k-1} = S_{2k-2} - \frac{1}{2} b_{2k-1} \quad \text{is decreasing} \quad \Rightarrow \quad S_{2k-1} < S_{2k} \leq S_1
\]

By monotone sequence theorem, \( S_{2k} \to S' \) and \( S_{2k-1} \to S'' \) have limits; and \( S' - S'' = \lim S_{2k} - \lim S_{2k-1} = \lim (S_{2k} - S_{2k-1}) = \lim (-b_{2k}) = 0 \). So \( S_n \to S'' \), and the series converges.

\[
\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \quad \text{is convergent by (10).}
\]

What is its sum? Write

\[
S_{2n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}
\]

\[
= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = \frac{1}{n} \left( \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right)
\]

\[
= \frac{1}{n} \left( \frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \cdots + \frac{1}{1+\frac{n}{n}} \right) = \frac{1}{n} \sum_{k=1}^{n} f(x_k) \quad f(x) = \frac{1}{x}
\]

\[
\Rightarrow \int_{\frac{1}{2}}^{1} \frac{dx}{x} = \ln(2). \quad \text{Riemann sum for} \int_{\frac{1}{2}}^{1} \frac{dx}{x}
\]

Problem: Which of \( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}, \sum_{n=1}^{\infty} (-1)^{n} \frac{3n+5}{n+1}, \)

\[1 - \frac{1}{4} + \frac{1}{3} - \frac{1}{16} + \frac{1}{5} - \frac{1}{36} + \frac{1}{7} - \frac{1}{64} + \frac{1}{9} - \cdots \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{n^2}{2^n} \quad \text{converge?} \]
Answer:

- C

- D - \( b_k \rightarrow 0 \)

- D - \( b_k \rightarrow 0 \) but not decreasing (can prove D)

- C - \( b_k \rightarrow 0 \) and \( b \) other 3x term!