Lecture 38: Power series

Last time we introduced the notion of uniform convergence (UC) of a function series \( \sum_{n=0}^{\infty} f_n(x) \) on a set \( S \). This means that for each \( \varepsilon > 0 \) we can find a \( N \) s.t. \( n \geq N \implies \left| \sum_{k=n+1}^{\infty} f_k(x) \right| < \varepsilon \) on \( S \). It guarantees that the sum is continuous if the \( f_n \) are, and that it may be integrated termwise. We showed that for a power series \( \sum_{n=0}^{\infty} a_n z^n \), there exists a unique \( r \in \mathbb{R}_{\geq 0} \) (the radius of convergence) so that:

- the series is AC on \( |z| < r \)
- the series is UC on \( |z| \leq R \), for any \( R < r \)
- the series diverges on \( |z| > r \)

We can also replace it everywhere by \( z-a \); then the disk is centered at \( a \) instead of 0. Some immediate consequences:

- \( f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n \) is continuous on \( (a-r, a+r) \)
- \( \int_a^x f(t) dt = \sum_{n=0}^{\infty} \frac{a_n (x-a)^{n+1}}{n+1} \) on \( (a-r, a+r) \) \( \implies \) radius of conv. of this is at least \( r \)

Proposition: Integration and differentiation termwise preserve the radius of convergence. Moreover, \( f(x) = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1} \).

Proof: Given \( a < x < y < a+r \), the MVT yields \( u_n \in (x, y) \) (for each \( n \)) s.t. \( (y-x)^n (u_n-a)^{-1} = n (y-a)^n \). Since \( \sum_{n=0}^{\infty} a_n (x-a)^n \), \( \sum_{n=0}^{\infty} a_n (y-a)^n \) AC, \( \frac{f(y)-f(x)}{y-x} = \sum_{n=0}^{\infty} a_n \frac{(y-x)^n}{y-x} = \sum_{n=0}^{\infty} n a_n (u_n-a)^{-1} \) is AC,

and so by comparison \( (|x-a| \leq |u_n-a|)^{-1} \sum_{n=1}^{\infty} n a_n (x-a)^{n-1} \) is AC.
Space $r$ can be taken arbitrarily close to $a$, the radius of convergence of \( \sum n_0 \frac{(x-a)^n}{n!} \) is at least $r$. So neither integration nor differentiation of series decreases radius of convergence, and \( \frac{1}{x-x_0} \) cannot change it. Moreover, \( \int_a^x g(x) \, dx = \sum \int_a^x \frac{(x-a)^n}{n!} \, dx = \sum_{n=1}^\infty \frac{a_n (x-a)^n}{n} = f(x) - a_0 \implies f'(x) = g(x). \)

**Ex:** Since the geometric series \( \frac{1}{1-x} = \sum_{n=0}^\infty x^n \) works on \( |x| < 1 \), integrals & derivatives work there too: hence \( -\log(1-x) = \sum_{n=1}^\infty \frac{x^n}{n} \) holds there, as well as the series for arctan \( x \) (by integrating \( \frac{1}{1+x^2} = \sum_{n=0}^\infty (-1)^n x^{2n} \)).

**Corollary:** If \( f(x) = \sum_{n=0}^\infty a_n (x-a)^n \) on \( (a-r, a+r) \), then \( f \) is infinitely differentiable there, and \( a_k = \frac{f^{(k)}(a)}{k!} \) for each \( k \).

**Proof:** Apply the Proposition \( k \) times and plug in \( x = a \). □

Conversely, if \( f \) is infinitely differentiable at \( a \), then we can form the Taylor series of \( f \) at \( a \): \( \sum_{n=0}^\infty \frac{f^{(n)}(a)}{n!} (x-a)^n \). If this equals \( f \) in some neighborhood of \( a \), then \( f \) is called an analytic function there. Unfortunately, not all smooth functions are analytic:

**Ex:** \( f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases} \) has \( f^{(n)}(0) = 0 \) \( (\forall n) \), so its Taylor series is \( 0 \). Obviously \( f \) isn't 0.

(See your HW.)

**Theorem:** If \( f \) is smooth on \( I = (a-R, a+R) \), and \( |f^{(n)}(x)| \leq B_n \) \( (x \in I) \) for some sequence \( \{B_n\} \) with \( \frac{B_n}{n!} \to 0 \) \( \forall x \in (0,R) \), then the Taylor series of \( f \) at \( a \) converges to \( f \) on \( I \):

\[
 f(x) = \sum_{n=0}^\infty \frac{f^{(n)}(a)}{n!} (x-a)^n \quad \forall x \in I.
\]
Proof: We need to show \( E_n(x) := f(x) - T_n(x) \to 0 \) for \( x \in \mathbb{T}. \)

Since \( E_n(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!} (x-t)^n \, dt \), writing \( R := |x - c| \) we have

\[
|E_n(x)| \leq \frac{B_{n+1}}{n!} \int_a^x (x-t)^n \, dt = \frac{B_{n+1} R^{n+1}}{(n+1)!} \to 0.
\]

\( \square \)

Ex/ \( |\sin(x)| \leq |x| \) \( \forall x \in \mathbb{R} \), and \( R^n \to 0 \) for any \( R \), so Thm. \( \Rightarrow \) \( \sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \). Some argument for \( \cos(x) \)

\( \Rightarrow \cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \). For \( f(x) = e^x \), \( f^{(n)}(x) = e^x \) on \((-R,R)\) is bounded by \( B_n = e^R \), and \( R^n e^R \to 0 \): so Thm. \( \Rightarrow \)

\( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \). All of these hold on any \((-R,R)\), so hold on \( \mathbb{R} \) — we already knew these series had \( \infty \) radius of convergence. //

Ex/ \( f(x) = (1+x)^\alpha \), \( \alpha \in \mathbb{R} \setminus \mathbb{Z} \) (i.e. \( \alpha \) not an integer).

We know that \( \frac{f^{(n)}(0)}{n!} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} = \binom{\alpha}{n} \), so the series is \( \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \). By the ratio test \( \rho = \lim_{n \to \infty} \left| \frac{(x+1)^{n+1}}{\binom{\alpha}{n+1} x^n} \right| = \lim_{n \to \infty} \left| \frac{x+1}{1+\frac{1}{x}} \right| = 1 \) \( \implies \) it is AC for \( |x| < 1 \)

\( (1+x)^\alpha \to 0 \) for \( |x| < 1 \). \( \implies \) \( n \binom{\alpha}{n} x^n \to 0 \).

Now going back to the proof above, we get

\( (\text{with } a = 0, \quad |x| < R < 1, \quad |E_{n,R}(x)| = \left| \int_0^x n \binom{\alpha}{n} (1+t)^{x-\alpha-1} (x-t)^n \, dt \right| \)

\( \leq n \left| \frac{\binom{\alpha}{n}}{n!} \right| \int_0^1 (1+t)^{x-\alpha-1} \, dt \leq n \left| \frac{\binom{\alpha}{n}}{n!} \right| x^{x-\alpha-1} \int_0^x (1+t)^{-x-1} \, dt \)

\( \text{(or } \int_0^x \text{ if } x < 0 \text{ maximum value of } |x| (1+t) = x \text{ at } t = 0) \)

\( \leq C n \left| \frac{\binom{\alpha}{n}}{n!} \right| x^n \to 0. \)

So \( f(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \) on any \((-R,R)\) with \( R < 1 \), hence on \((-1,1)\). //
Application to differential equations

How to solve \( x(1-x) y'' + (1-2x) y' - \frac{2}{q} y = 0 \) in a neighborhood of 0? Try formally substituting \( y = \sum_{n=0}^\infty a_n x^n \); we get (for all \( x \) in some interval of convergence)

\[
0 = x(1-x) \sum_{n=2}^\infty n(n-1)a_n x^{n-2} + (1-2x) \sum_{n=1}^\infty n a_n x^{n-1} - \frac{2}{q} \sum_{n=0}^\infty a_n x^n
\]

\[
= \sum_{n=2}^\infty n(n-1)a_n x^{n-1} - \sum_{n=2}^\infty n(n-1)a_n x^{n-1} + \sum_{n=1}^\infty n a_n x^{n-1} - \sum_{n=1}^\infty 2n a_n x^n - \sum_{n=0}^\infty \frac{2}{q} a_n x^n
\]

\[
= \sum_{n=1}^\infty (n^2 - n)a_n x^n - \sum_{n=2}^\infty n(n-1)a_n x^{n-1} + \sum_{n=1}^\infty (n+1)a_n x^n - \sum_{n=1}^\infty 2n a_n x^n - \sum_{n=0}^\infty \frac{2}{q} a_n x^n
\]

\[
= (a_1 - \frac{2}{q} a_0) + (4a_2 - \frac{10}{q} a_1) x + \sum_{n=2}^\infty \left( \frac{(n+2n+1)a_{n+1} - (n^2 + n + \frac{2}{3})a_n}{(n+3)(n+\frac{3}{2})} \right) x^n.
\]

Taking \( a_0 = 1 \), we get \( a_1 = \frac{2}{q} \), \( a_2 = \frac{10}{81} \), \( a_{n+1} = \frac{(n+1)(n+\frac{3}{2})}{(n+3)(n+\frac{3}{2})} a_n \)

\[
= \frac{(\frac{1}{2} \cdot \frac{4}{3} \cdots \frac{3n-2}{3})}{(n!)^2} \frac{(\frac{2}{3} \cdot \frac{5}{3} \cdots \frac{3n}{3})}{(n!)^3} = \frac{1}{(\frac{1}{2} \cdot \frac{3}{2} \cdots \frac{3n}{3})(\frac{2}{3} \cdot \frac{5}{3} \cdots \frac{3n-1}{3})} = \frac{1}{(\frac{1}{2} \cdot \frac{3}{2} \cdots \frac{3n}{3})(\frac{2}{3} \cdot \frac{5}{3} \cdots \frac{3n-1}{3})}
\]

\[
= \frac{(3n)!}{(n!)^3 3^n}, \quad \text{and so}
\]

\[
f(x) = \sum_{n=0}^\infty \frac{3n!}{n!^3 3^n} (\frac{x}{27})^n \quad \text{which has radius of convergence}
\]

\[
r = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \to \infty} \frac{(n+1)^3 3^3}{(3n+1)(3n+2)(3n+3)} = \lim_{n \to \infty} \frac{(n+1)^3}{(n+1)(n+\frac{5}{3})(n+\frac{7}{3})} = 1.
\]

This is an example of a "hypergeometric function", a new type of function discovered by Gauss that can't be written in terms of the ones you already know.