Lecture 38: Power series

Last time we introduced the notion of uniform convergence \((UC)\) of a function series \(\sum_{n=0}^{\infty} f_n(x)\) on a set \(S\). This means that for each \(\varepsilon > 0\) we can find an \(N\) s.t. \(n \geq N \Rightarrow \sum_{k=n+1}^{\infty} |f_k(x)| < \varepsilon\) on \(S\).

It guarantees that the sum is continuous if the \(\{f_n\}\) are, and that it may be integrated termwise. We showed that for a power series \(\sum_{n=0}^{\infty} a_n z^n\), there exists a unique \(r \in \mathbb{R}_+ \cup \{\infty\}\) (the radius of convergence) so that:

- the series is \(AC\) on \(|z| < r\)
- the series is \(UC\) on \(|z| \leq R\), for any \(R < r\)
- the series diverges on \(|z| > r\) \(\text{ or } z < -r\)

We can also replace it everywhere by \(z-a\); then the disk is centered at \(a\) instead of \(0\). Some immediate consequences:

- \(f(x) := \sum_{n=0}^{\infty} a_n (x-a)^n\) is continuous on \((a-r, a+r)\)
- \(\int_a^x f(t) \, dt = \sum_{n=0}^{\infty} \frac{a_n (x-a)^{n+1}}{n+1}\) on \((a-r, a+r)\) \(\Rightarrow \text{ radius of conv. of this}\) is at least \(r\)

Proposition: Integration and differentiation termwise preserve the radius of convergence. Moreover, \(f'(x) = \sum_{n=1}^{\infty} na_n (x-a)^{n-1}\).

Proof: Given \(a < x < y < a + r\), the MVT yields \(u_n \in (x, y)\) (for each \(n\)) s.t. \((y-x)(x-a)^n = n (u_n-a)^{n-1}\). Since \(\sum a_n (x-a)^n\), \(\sum a_n (y-a)^n\) \(AC\),

\[
\frac{f(y)-f(x)}{y-x} = \sum_{n=0}^{\infty} a_n \frac{(y-a)^n - (x-a)^n}{y-x} = \sum_{n=0}^{\infty} na_n (u_n-a)^{n-1} \quad \text{is AC,}
\]

and so by comparison \((|x-a| \geq |a_n-a|)^{n-1}\) \(\sum n a_n (x-a)^{n-1} \quad \text{is AC}.

This is valid for \(x \rightarrow a^+\) and \(y \rightarrow a^-\).
Since \( r \) can be taken arbitrarily close to \( a + r \), the radius of convergence of \( \sum a_n(x-a)^n \) is at least \( r \). So neither integration nor differentiation of series decreases radius of convergence, and \( \int \) neither can change it. Moreover, \( \int f(x) \, dx = \sum \int a_n(x-a)^n \, dx = \sum a_n(x-a)^n = f(x) - a_0 \Rightarrow f'(x) = g(x) \).

**Example:** Since the geometric series \( \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \) works on \( |x| < 1 \), integral & derivative work there too: here \( -\log(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \) holds true, as well as the series for \( \sin(x) \) (by integrating \( \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \)).

**Corollary:** If \( f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n \) on \((a-r, a+r)\), then \( f \) is infinitely differentiable there, and \( a_k = \frac{f^{(k)}(a)}{k!} \) for each \( k \).

**Proof:** Apply the Proposition \( k \) times and plug in \( x = a \).

Conversely, if \( f \) is infinitely differentiable at \( a \), then we can form the Taylor series of \( f \) at \( a \): \( \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \). If this equals \( f \) in some neighborhood of \( a \), then \( f \) is called an analytic function there. Unfortunately, not all smooth functions are analytic:

**Example:** \( f(x) := \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases} \) has \( f^{(n)}(0) = 0 \) \( (\forall n) \), so its Taylor series is \( 0 \). Obviously \( f \neq 0 \).

(See your HW.)

**Theorem:** If \( f \) is smooth on \( I = (a+R, a-R) \), and \( |f^{(n)}(x)| \leq B_n (4\pi x^4) \) for some sequence \( \{B_n\} \) with \( \frac{B_n}{n!} \to 0 \) \( \forall x \in (0,R) \), then the Taylor series of \( f \) at \( a \) converges to \( f \) on \( I \):

\[ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \quad \forall x \in I. \]
Proof: We need to show \( E_n(x) = f(x) - T_n(x) \to 0 \) for \( x \in \mathbb{S} \).

Since \( E_n(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \), writing \( R = |x-a| \) we have

\[
|E_n(x)| \leq \frac{B_{n+1}}{n!} \int_{a}^{x} |x-t|^n dt = \frac{B_{n+1} R^{n+1}}{(n+1)!} \to 0.
\]

\( \square \)

Ex. \( |\sin n(x)| \leq 1 \) \( \forall x \in \mathbb{R} \), and \( \frac{R^n}{n!} \to 0 \) for any \( R \), so Thm. \( \lim \frac{\sin(x)}{x} = 1 \). Some argument for 

\( \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \). For \( f(x) = e^x \), \( \frac{f^{(n)}(x)}{n!} = e^x \) on 

\((-R, R)\) is bounded by \( B_n = e^R \), and \( \frac{R^ne^R}{n!} \to 0 \); so Thm. \( \Rightarrow \)

\( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \). All of these hold on any \((-R, R)\), so hold on \( \mathbb{R} \) — we already knew these series had \( n \to 0 \) radius of convergence.

Ex. \( f(x) = (1+x)^{\alpha} \), \( \alpha \in \mathbb{R} \) (i.e. \( \alpha \) not an integer).

We know that \( \frac{f^{(n)}(0)}{n!} = \frac{\alpha(\alpha-1) \cdots (\alpha-n+1)}{n!} = \binom{\alpha}{n} \), so

the series is \( \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \). By the ratio test \( (\rho = \lim_{n \to \infty} \frac{|(\alpha+n)!|}{(\alpha+n+1)!} \lim_{n \to \infty} \frac{|x| \cdot |1|}{|x||1|} = \lim_{n \to \infty} \frac{|x|}{n+1} = |x| = |x| < 1 \Rightarrow (\alpha)x^n \to 0 \) for \( |x| < 1 \) (\( \Rightarrow n(\alpha)x^n \) also \( \to 0 \)).

Now going back to the proof above, we get

\[
(\text{with } a=0, \ |x| < R < 1) \quad |E_n(x)| = \left| \int_{0}^{x} n(\binom{x}{n}) (1+t)^{\alpha-n}(x-t)^{n-1} \, dt \right|
\]

\[
\leq n \left| (\binom{x}{n}) \right| \left( \int_{x=0}^{x=1} \frac{|x-t|^{n-1}}{|1+t|} \, dt \right) = n \left| (\binom{x}{n}) \right| |x|^{n-1} \int_{0}^{1} \left( 1 + x^{-1} \right) \, dt
\]

\[
= n \left| (\binom{x}{n}) \right| |x|^{n-1} \int_{0}^{\infty} \left( 1 + x^{-1} \right) \, dx
\]

\[
\text{or } \int_{x=0}^{x=1} \text{ maximum value of } |x| |1 + x^{-1}| \leq C n \left| (\binom{x}{n}) \right| |x|^{n-1} \to 0.
\]

So \( f(x) = \sum_{n=0}^{\infty} \binom{x}{n} x^n \) on any \((-R, R)\) with \( R < 1 \), hence on \((-1, 1)\).
Application to differential equations

How to solve \( x(1-x)y'' + (1-2x)y' - \frac{2}{9}y = 0 \) in a neighborhood of 0.² Try formally substituting \( y = \sum_{n=0}^{\infty} c_n x^n \); we get (for all \( x \) in some interval of convergence)

\[
0 = x(1-x) \sum_{n=1}^{\infty} n(n-1)c_n x^{n-2} + (1-2x) \sum_{n=1}^{\infty} nc_n x^{n-1} - \frac{2}{9} \sum_{n=0}^{\infty} c_n x^n
\]

\[
= \sum_{n=2}^{\infty} n(n-1)c_n x^{n-1} - \sum_{n=1}^{\infty} n(n-1)c_n x^n + \sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=1}^{\infty} 2nc_n x^n - \sum_{n=0}^{\infty} \frac{2}{9} c_n x^n
\]

\[
= \sum_{n=1}^{\infty} (n+1)c_{n+1} x^n - \sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=1}^{\infty} (n+1)c_n x^n - \sum_{n=1}^{\infty} 2nc_n x^n - \sum_{n=0}^{\infty} \frac{2}{9} c_n x^n
\]

\[
= (a_1 - \frac{2}{9}a_0) + (4a_2 - \frac{20}{9}a_1) x + \sum_{n=2}^{\infty} \left( \frac{(n+2n+1)a_{n+1} - (n^2+n+\frac{2}{3})a_n}{(n+1)!} \right) x^n .
\]

Taking \( a_0 = 1 \), we get \( a_1 = \frac{2}{9} \), \( a_2 = \frac{10}{81} \), \( a_{n+1} = \frac{(n+1)(n+\frac{3}{2})}{(n+1)!} \) \( a_n \)

\[
= \frac{(\frac{n}{3})!}{(\frac{n}{3})!3^{n/3}}, \quad \text{and so}
\]

\[
f(x) = \sum_{n=0}^{\infty} \frac{3n!}{n!^3} \left( \frac{x}{27} \right)^n \quad \text{which has radius of convergence}
\]

\[
r = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left( \frac{3n+1}{3n+3} \right)^n \left( \frac{3n+2}{3n} \right)^{3n} = \lim_{n \to \infty} \frac{(n+1)^3}{(n+1)(n+\frac{3}{2})(n+\frac{5}{2})} = 1.
\]

This is an example of a "hypergeometric function", a new type of function discovered by Gauss that can't be written in terms of the ones you already know.