Lecture 40: Linear independence

Let's begin by doing some calculations.

Example: 
\[ A = (2, -5, -1), \quad B = (-7, -4, 6) \]

\[ A \cdot A = 30 \Rightarrow ||A|| = \sqrt{30}, \quad B \cdot B = 101 \Rightarrow ||B|| = \sqrt{101} \]

\[ A \cdot B = 0 \Rightarrow A \perp B \]

We can produce unit vectors in the directions of \( A \) and \( B \) by:

\[ \frac{A}{||A||} = \frac{1}{\sqrt{30}} (2, -5, -1), \quad \frac{B}{||B||} = \frac{1}{\sqrt{101}} (-7, -4, 6). \]

(Works since \( \frac{A}{||A||} \cdot \frac{B}{||B||} = \frac{A \cdot A}{||A||^2} \frac{B \cdot B}{||B||^2} = 1 \).)

The distance between \( A \) and \( B \) (as points) is:

\[ ||A - B|| = ||(9, -1, -4)|| = \sqrt{131}. \]

Projections

Going down a dimension to \( V_2 \), we want to calculate the projection of \( A \) onto \( B \), as shown.

To do this, write \( A = C + tB \), where \( C \cdot B = 0 \).

This gives:

\[ A \cdot B = C \cdot B + tB \cdot B = tB \cdot B \Rightarrow t = \frac{A \cdot B}{B \cdot B} = \frac{A \cdot B}{||B||^2} \]

\[ \Rightarrow tB = \left( \frac{A \cdot B}{||B||^2} \right) \frac{B}{||B||}. \]

More generally, these formulas are taken as the definitions of projection and angle in higher dimensions.

by Cauchy-Schwarz, this is always between \(-1, 1\), so is cosine of something! \]
Ex / A = (1, 2, 1), B = (1, 1, 0) \Rightarrow \cos \theta = \frac{(1, 2, 1) \cdot (1, 1, 0)}{\|1, 2, 1\| \|1, 1, 0\|} = \frac{3}{\sqrt{5} \sqrt{3}} = \frac{\sqrt{3}}{2}
\Rightarrow \theta = \frac{\pi}{6} \text{ (always taken to be in } [0, \pi])

The law of cosines is an immediate consequence of our definition of angle: $\|A - B\|^2 = \|A\|^2 + \|B\|^2 - 2\|A\|\|B\| \cos \theta$.

Linear Combinations

If $A_1, \ldots, A_n \in \mathbb{V}_n$, a linear comb. of them is $\sum c_i A_i \ (c_i \in \mathbb{R})$. Their span (or linear span) is the set of all of these linear combinations. If this span equals $\mathbb{V}_n$, we say "the $\{A_i\}$ span $\mathbb{V}_n$.

Ex / $\{E_1, E_2, \ldots, E_n\}$ span $\mathbb{V}_n$, where $E_i := (0, \ldots, 0, 1, 0, \ldots, 0)$ are the "unit coordinate vectors". Any $A$ can be written as a linear combination of them (in a unique way, as it turns out):

$A = (a_1, a_2, \ldots, a_n) = a_1 (1, 0, \ldots, 0) + a_2 (0, 1, 0, \ldots, 0) + \cdots + a_n (0, 0, \ldots, 0, 1)$

$= \sum_{i=1}^n a_i E_i$.

Note: If $S = \{A_1, \ldots, A_n\}$, write $L(S)$ for the linear span.

$S$ spans $\mathbb{V}_n \Leftrightarrow L(S) = \mathbb{V}_n$.

Definition: A set $S$ of vectors is linearly independent if $\sum_{i=1}^n c_i A_i = 0 \Rightarrow \text{ all } c_i = 0$.

(The basis calls this "spanning the 0-vector uniquely"). An immediate consequence is that $S$ spans every vector in $L(S)$ uniquely;
the same that \( \sum c_i A_i = \sum d_i A_i \Rightarrow c_i = d_i \) (V.i).

\[
\begin{align*}
\text{Ex:} & \quad S = \{(1,2), (1,0), (0,1)\} \text{ spans } V_2 \text{ but is not independent} \\
S = \{(1,0)\} \text{ doesn't span } V_2 \text{ but is independent} \\
S = \{(1,0), (1,2)\} \text{ spans } V_2 \text{ and is independent}
\end{align*}
\]

Note: if a set contains the 0-vector, it is not independent. (V.i).

**Definition:** \( S \) is a basis of \( V_n \) if it spans \( V_n \) and is linearly independent.

**Theorem:**
(i) Any basis of \( V_n \) consists of exactly \( n \) vectors.
(ii) Any set of linearly independent vectors is a subset of a basis.

**Corollary:** Any set of \( n \) linearly independent vectors is a basis.

To prove the Theorem, we'll use the following

**Lemma:** If \( S = \{A_1, \ldots, A_k\} \subset V_n \) is linearly independent, then any set of \( k+1 \) vectors in \( L(S) \) is dependent.

**Proof:** (Base case) \( S = \{A_1\}, A_1 \neq 0, L(S) \) consists of multiples of \( A_1 \), so any two are dependent (\( B_i = c_i A_1 \Rightarrow c_i B_2 - c_i B_1 = 0 \)) (inductive step) \( T = \{B_1, \ldots, B_{k+1}\} \subset L(S), B_i = \sum a_{ij} A_j \).

If \( a_{i1} = 0 \) (V.i), then \( T \subset L(\{A_2, \ldots, A_k\}) \Rightarrow T \) dependent.

Otherwise, since \( a_{i1} \neq 0 \), we may assume \( a_{i1} \neq 0 \).

Writing \( c_i = \frac{a_{i1}}{a_{i1}}, \quad c_i B_1 = a_{i1} A_1 + \sum_{j \neq i} c_i a_{ij} A_j \)

\[
\begin{align*}
-(B_i = a_{i1} A_1 + \sum_{j \neq i} a_{ij} A_j) \\
c_i B_1 - B_i = \sum (c_i a_{ij} - a_{ij}) A_j
\end{align*}
\]

\( \Rightarrow \{c_i B_1 - B_i\} \subset L(\{A_2, \ldots, A_k\}) \Rightarrow \{c_i B_1 - B_i\} \) dependent.
\[ \sum_{i=2}^{k} t_i (c_i B_i - B_i) = 0 \quad \text{for all } c_i \in \mathbb{R} \implies \left( \sum_{i=2}^{k} t_i c_i \right) B_2 - \sum_{i=2}^{k} t_i B_i = 0 \implies T \text{ is dependent.} \]

**Proof of Theorem:**

(i) \( V_n = L(\{E_1, \ldots, E_n\}) \). By the lemma, if \( S \) consists of more than \( n \) vectors, it is dependent (hence not a basis). If \( S \) consists of less than \( n \) vectors, it can't span \( V_n \) — otherwise, the lemma would say \( E_1, \ldots, E_n \) are dependent. So if \( S \) is to be a basis, it had better consist of exactly \( n \) vectors.

(ii) \( S = \{A_1, \ldots, A_n\} \) independent. If it doesn't span \( V_n \), pick \( A_k \) such that \( \mathbf{v} \notin L(S) \). \( \{A_1, \ldots, A_{k-1}\} \) is still independent (why?). Continue in this fashion until you have \( n \) elements. At this point the set spans \( V_n \) — otherwise we'd get \( n+1 \) independent vectors at the next step, which is absurd by the lemma (since \( \{E_1, \ldots, E_n\} \) spans \( V_n \)).

**Ex:** Let \( S = \{A_1, \ldots, A_n\} \) be orthonormal, i.e., \( A_i \cdot A_j = 0 \quad \forall i \neq j \).

Then \( S \) is independent: if \( 0 = \sum_{i=1}^{n} c_i A_i \) then taking \( A_j \cdot \) both sides gives \( 0 = c_j \|A_j\|^2 = c_j \) (\( \forall j \)). (In particular, if \( k = n \) then \( S \) is a basis of \( V_n \).) For any \( B \in L(S) \), we have \( B = \sum_{i=1}^{n} b_i A_i \implies A_j \cdot B = b_j A_j \cdot A_j \implies b_j = \frac{A_j \cdot B}{A_j \cdot A_j} \)

\[ B = \sum_{i=1}^{n} \frac{B \cdot A_i}{A_i \cdot A_i} A_i \] is the unique linear combination of \( \{A_i\} \) giving \( B \).