Lecture 41: Lines & planes in n-space

**LINES**

Given \( P \in V_n\), \( A \in V_n^{\mathbb{R}^{n+1}} \), set \( L(P, A) = \{ P + tA \mid t \in \mathbb{R} \} \subset V_n \).

This is the linear span \( L(A) \) translated by \( P \) : \( L(P, A) = L(A) + P \).

It is parametrized by \( (= \) the image of \( X \))

\[
    \text{(parametric form)} \quad X : \mathbb{R} \rightarrow V_n \\
    t \mapsto X(t) = (p_1, \ldots, p_n) + t(a_1, \ldots, a_n) = (p_1 + ta_1, \ldots, p_n + ta_n)
\]

**Facts:**

- A line \( l \) contains \( P \) \( \iff \) \( l = L(P, A) \) for some \( A \).

**Proof:** \((\Leftarrow)\) obvious

\((\Rightarrow)\) Given \( l = L(Q, B) \ni P \), write \( l' = L(P, B) \). I claim \( l = l' \).

Indeed, \( P \in l \Rightarrow P = Q + t_0B \). So \( X \in l' \iff X = Q + t_xB \Rightarrow X = (P - t_0B) + t_xB = P + (t_x - t_0)B = P + t_x'B \Rightarrow X \in l' \). \( \square \)

- \( L(P, A) = L(P, B) \iff A \parallel B \).

**Definition:** \( L(P, A) \parallel L(Q, B) \iff A \parallel B \).

- If \( P \neq Q \), then \( L(P, Q-P) \) is the unique line containing \( P \& Q \). (Reduce to the case \( P = 0 \); then a line containing \( Q \) contains \( L(Q) \) hence \( L(Q) \).)

- \( \{A, B\} \subset V_n \) is linearly independent \( \iff \) \( A, B \) don't lie on a line through \( 0 \) \( \iff \) \( A, B \neq 0 \) and \( A \not\parallel B \) not parallel.

**Proof:** \((\Rightarrow)\) If they lie on a line \( l \) through \( 0 \), then \( l = L(0, C) = L(C) \Rightarrow A = t_0C, B = t_1C \Rightarrow t_0A - t_1B = 0 \Rightarrow \) dependent (not L.I.)

\((\Leftarrow)\) If \( A, B \) dependent, then \( B = cA \) (or vice versa) \( \iff \) both in \( L(0, A) \).

**Ex.** Do \( P = (2, 1, 1), Q = (4, 1, -1), R = (3, -1, 1) \) lie on a straight line?

(Hint: equiv. to same question for \( 0, Q-P, R-P \).)
Ex. \((n=2)\) Let \(l=(P, A)\), \(A=\langle a_1, a_2 \rangle\); and consider \(N:=\langle a_2, -a_1 \rangle\). Since \(N \cdot A = 0\), \(N \perp A\).

Let \(I := \{X \in \mathbb{R}_2 : (X-P) \cdot N = 0\}\). \(\square\) (Cartesian form)

Then \(l = I\) because \(X \in I \iff X = P + \epsilon A \iff X - P = \epsilon A \iff (X-P) \cdot N = 0\).

Now write \(\text{proj}_N(P) := (P \cdot \frac{N}{\|N\|}) \frac{N}{\|N\|} \in L(N)\). This is on \(l\) because \((\frac{P \cdot N}{N \cdot N} N - P) \cdot N = \frac{P \cdot N}{N \cdot N} N - P \cdot N = 0\). We claim that \(\|\text{proj}_N(P)\| = \frac{P \cdot N}{\|N\|}\) is the distance from \(O\) to \(l\):

- **Cauchy-Schwarz proof:** \(\|X\| = \frac{\|X\|\|N\|}{\|N\|} \geq \frac{|X \cdot N|}{\|N\|} = \frac{|P \cdot N|}{\|N\|}\) for any \(X \in l\) \(\square\)

- **Calculus proof:** \[ \frac{d}{dt} |X(t)|^2 = \frac{d}{dt} X(t) \cdot X(t) = 2 \dot{X}(t) \cdot X(t) \]
  \((X(t) = P + tA)\)
  \[\Rightarrow 2A \cdot X(t)\]. If \(t = t_0\) solves this, then
  \[X(t_0) \perp A \Rightarrow X(t_0) \parallel N \Rightarrow X(t_0) = L(N) \cap l = \text{proj}_N(P)\]. \(\square\)

More generally, if we want \(\text{dist}(Q, l)\) for \(Q\) different from \(O\), just translate the line to \(l - Q = L(P-Q, A)\), so that

\[\text{dist}(Q, l) = \text{dist}(O, L(P-Q, A)) = \|\text{proj}_N(P-Q)\| = \frac{|P-Q \cdot N|}{\|N\|}\]. That is, you take any point \(P \in l\) and \(\text{dist}(P-Q)\) with the unit normal vector \(\frac{N}{\|N\|}\). \(\square\)
**Planes**

Given $P \in \mathbb{V}_n$, $\{A, B\} \subset \mathbb{V}_n$ independent, set

$$M(P, \{A, B\}) := \{ P + sA + tB \mid s, t \in \mathbb{R} \} = L(\{A, B\}) + P.$$  

A plane is any subset of $\mathbb{V}_n$ of this form; we may regard it as the image of a parameterization map

$$X : \mathbb{R}^2 \rightarrow \mathbb{V}_n$$

$$(s, t) \mapsto X(s, t) := P + sA + tB = (p_1 + sa_1 + tb_1, \ldots, p_n + sa_n + tb_n).$$

Two planes $M(P, \{A, B\})$, $M(Q, \{C, D\})$ are called parallel if $L(\{A, B\}) = L(\{C, D\})$.

**Facts:**

- $M(P, \{A, B\}) = M(P, \{C, D\}) \iff L(\{A, B\}) = L(\{C, D\})$ [just use (*)]
- $M(P, \{A, B\}) = M(Q, \{A, B\}) \iff Q \in M(P, \{A, B\})$
  
  [again, (*) gives $L(\{A, B\}) \subseteq L(\{A, B\}) + P = L(\{A, B\}) + Q \iff Q - P \in L(\{A, B\})$]
- If $P, Q, R$ aren't collinear (don't lie on a line), then $O, Q - P, R - P$ are non-collinear $\implies Q - P \& R - P$ are not parallel $\implies \{Q - P, R - P\}$ independent.
  
  So we may define $M_{PQR} := M(P, \{Q - P, R - P\})$, which contains $P, Q, R$.

**Theorem:** $M_{PQR}$ is the only plane containing $P, Q, R$. ($\Rightarrow$ non-collinear pts. determine a plane)

**Proof:** $\{\text{planes containing } P, Q, R\} = \{\text{planes containing } O, Q - P, R - P\} + P$.

So we may assume $P = O$. We need then to show that $M := L(\{Q, R\})$ is the only plane containing $O, Q, R$, given $\{Q, R\}$ linearly independent.

Let $M' := M(O, \{A, B\}) = L(\{A, B\})$ also contain $O, Q, R$.

Then $Q = aA + bB$ & $R = cA + dB$ $\implies L(\{B, Q\}) \subseteq L(\{A, B\})$.

$2bQ = a(2A + bB)$ & $6cR = 6cA + 6dB$ $\implies aQ - cR = (ac - bc)A$
\[ A = \frac{d}{ad-bc} \mathbf{Q} + \frac{-b}{ad-bc} \mathbf{R} \in \mathbf{L}(\mathbb{R}^2). \] (We knew \( ad-bc \neq 0 \); otherwise \( d\mathbf{Q} - b\mathbf{R} = 0 \Rightarrow \{ \mathbf{Q}, \mathbf{R} \} \) dependent.) Similarly, \( B \in \mathbf{L}(\mathbb{R}^2) \); \( d \) \( \therefore \) \( \mathbf{L}(\{A, B\}) \subseteq \mathbf{L}(\{R, Q\}) \). So \( M' = M \), done. \( \square \)

**Corollary:** 3 vectors in \( V_n \) are dependent \( \Rightarrow \) they lie on the same plane than the origin \( \mathbf{O} \).

**Proof:** \((\Rightarrow)\) \( \forall \mathbf{C} : \mathbf{C} = a\mathbf{A} + b\mathbf{B} \Rightarrow \mathbf{C} \in \mathbf{L}(\{A, B\}) \)\( \uparrow \)

\((\Leftarrow)\) \( \forall \mathbf{A}, \mathbf{B} \) independent; then \( \exists \) plane than \( \mathbf{O} \) (namely \( \mathbf{L}(\{A, B\}) \) containing \( \mathbf{A}, \mathbf{B} \) (by Thm.). Hence \( \mathbf{C} \in \mathbf{L}(\{A, B\}) \). \( \square \)

**Ex**/

Consider the plane \( \mathbf{M} = \mathbf{M}(1, 1, -1), \{(2, 2, 3), (2, -2, -1)\} \subset \mathbf{V}_3 \).

Which of the points \((2, 2, 3), (4, 0, -2), (5, 1, -3), (3, 1, 3), (0, 0, 0)\) lie on \( \mathbf{M} \)? (Hint: Write \( \mathbf{M} \) parametrically, then eliminate \( \mathbf{C} \) to get a Cartesian equation.) \( \uparrow \)

Next time, we will do the plane analogue of the normal vector business (done above for lines). The key point is to introduce the cross-product to define \( \mathbf{N} \) by \( \mathbf{N} := (\mathbf{R} - \mathbf{P}) \times (\mathbf{Q} - \mathbf{P}) \). This allows us to (for example) find the distance from a point to a plane.

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† **WARNING:** \( \{A, B, C\} \) dependent does not imply that each one is a linear combination of the others. It only implies that at least one of them is a linear combination of the other two. (Why?)