Lecture 42: The cross-product

We will work entirely in \( V_3 \) for this lecture. Let \( \begin{align*} A &= (a_1, a_2, a_3) \\ B &= (b_1, b_2, b_3) \\ C &= (c_1, c_2, c_3) \end{align*} \)

Definition: (i) \( B \times C := (b_2c_3 - b_3c_2, b_3c_1 - b_1c_3, b_1c_2 - b_2c_1) = \begin{vmatrix} E_1 & E_2 & E_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \)

determinant notation (will review in class)

where \( E_1 = (1,0,0) \), \( E_2 = (0,1,0) \), \( E_3 = (0,0,1) \) aresometimes called \( i, j, k \). It's clear at once that \( C \times B = - B \times C \).

(ii) \( A \cdot (B \times C) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \) is the scalar triple-product.

more compact notation: \([ABC] \)

From this it's immediate that \( B \cdot (B \times C) = 0 = C \cdot (B \times C) \), so \( B \times C \) is \( \perp \) to \( B \) & \( C \).

Theorem 1: (i) \( \| A \times B \| = \| A \| \| B \| \| \sin \theta \| \)(where \( \theta \) is the angle between \( A \& B \))

= area of parallelogram spanned by \( A \& B \)

(ii) \( |A \cdot (B \times C)| \) = volume of parallelepiped spanned by \( A, B, C \)

Proof: (i) \( \| A \times B \|^2 + (A \cdot B)^2 = (A \times B) \cdot (A \times B) + (\sum a_i b_i)^2 \)

\[ \begin{align*} &= \sum_{i<j} (a_i b_j - a_j b_i)^2 + \sum_i a_i^2 b_i^2 + \sum_{i<j} a_i b_i a_j b_j \\ &= \sum_{i<j} a_i^2 b_j^2 + \sum_{i<j} a_j^2 b_i^2 - 2 \sum_{i<j} a_i b_i a_j b_j + 2 \sum_i a_i^2 b_i^2 \\ &= \sum_{i<j} a_i^2 b_j^2 + \sum_i a_i^2 b_i^2 \end{align*} \]

\( \sum_{i<j} a_i^2 b_j^2 + \sum_i a_i^2 b_i^2 = \sum_i \sum_{j \neq i} (a_i b_j)^2 = (\sum_i a_i^2)(\sum_j b_j^2) = \| A \|^2 \| B \|^2 \)

Now take square root.

\[ \| A \times B \|^2 = \| A \|^2 \| B \|^2 - \| A \|^2 \| B \|^2 \sin^2 \theta \]
The volume of the solid, since the cross-sectional area is constant (= area(base)) is 
\[ \text{area(base)} \cdot \text{height} = \frac{1}{2} |B \times C| \cdot |A| \cos \theta = \frac{1}{2} |A \cdot (B \times C)|. \]

Since the parallelepiped doesn't depend on the order of the vectors, we see that
\[ A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B) \quad (= (A \times B) \cdot C). \]

(OK, there is a question of what the sign is: either appeal to the "right-hand rule" or wait until we learn more about determinants.)

Before proceeding to the next result, recall that for the proof of the Cauchy-Schwarz inequality \[ |A \cdot B| \leq |A||B| \] we used
\[ 0 \leq \| B \| A - \frac{A \cdot B}{\|B\|} B \|^2 = \| A \|^2 - (A \cdot B)^2. \]
Therefore (for \( A \neq 0 \)) we have equality in Cauchy-Schwarz if and only if
\[ \| B \| A = \frac{A \cdot B}{\|B\|} B \iff A \parallel B. \]

**Theorem 2**: (i) \( \{A, B\} \) independent \( \iff \) \( A \times B \neq 0 \) \( \iff \{A, B, A \times B\} \) independent.

(ii) Assume \( \{A, B\} \) l.i. Then \( N \perp A, B \) \( \iff \) \( N \parallel A \times B \)

(iii) \( \{A, B, C\} \) independent \( \iff \) \( A \cdot (B \times C) \neq 0 \)

**Proof**: (i) By Theorem 1(ii), \( \| A \times B \| = \| A \| \| B \| \sin \theta \) is zero \( \iff \) \( A \) or \( B \) is \( 0 \) or \( A \parallel B \) \( \iff \) \( \{A, B\} \) dependent.

Now suppose \( \{A, B\} \) independent, so that \( \| A \times B \| \neq 0 \), and write \( aA + bB + c(A \times B) = 0. \) Dotting with \( A \times B \) gives
\[ 0 + 0 + c \| A \times B \|^2 = 0 \quad \Rightarrow \quad c = 0 \]
\[ aA + bB = 0 \quad \Rightarrow \quad a = b = 0. \] So \( \{A, B, A \times B\} \) are \( A, B \) independent.
(ii) Since \( \{A, B, A \times B\} \) L.I. by (i), they are a basis of \( V_3 \).

So

\[
N = \alpha A + \beta B + \gamma (A \times B)
\]

\[
\Rightarrow \|N\|^2 = 0 + 0 + c \|A \times B\|^2
\]

\[
\Rightarrow \|N\|^2 = 0 + 0 + c N \cdot (A \times B)
\]

\[
\Rightarrow \|N\|^2 \|A \times B\|^2 = (N \cdot (A \times B))^2 \|A \times B\|^2 = (N \cdot (A \times B))^2 (N \cdot (A \times B))
\]

\[
\Rightarrow \|N\| \|A \times B\| = |N \cdot (A \times B)| \quad \text{is equality in Cauchy-Schwarz}
\]

\[
\Rightarrow N \parallel A \times B.
\]

(iii) (\(\Longleftrightarrow\)): Suppose \( \{A, B, C\} \) dependent. Then either

- \( \{B, C\} \) dependent \(\Rightarrow (2)\) \(B \cdot C = 0\)
- \( \{B, C\} \) independent \& \(A = b B + c C \Rightarrow A \cdot (B \times C) = 0\)

(since \(B, C \perp B \times C\))

\(\Rightarrow\) : Suppose \( A \cdot (B \times C) = 0\). If \(B \times C = 0\), then

\(\{B, C\} \) dependent by (i) \(\Rightarrow \{A, B, C\} \) dependent, done.

If \(B \times C \neq 0\), then \(\{B, C\} \) & \(\{B, C, B \times C\}\) basis

\(\Rightarrow A = \alpha (B \times C) + \beta B + \gamma C\)

\(\Rightarrow A \cdot (B \times C) = \alpha \|B \times C\|^2 + 0 + 0 \Rightarrow \alpha = 0\)

\(\Rightarrow A = b B + c C \Rightarrow \{A, B, C\} \) dependent. \(\Box\)

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**Application to linear systems.** Consider

\[
\begin{align*}
\begin{cases}
\alpha \bar{x} + \beta y + \gamma z &= d_1, \\
\eta \bar{x} + \beta \bar{y} + \gamma \bar{z} &= d_2, \\
\zeta \bar{x} + \beta \bar{y} + \gamma \bar{z} &= d_3
\end{cases}
\end{align*}
\]

with \(\{A, B, C\}\) independent.

Then there exists a unique solution in \((\bar{x}, \bar{y}, \bar{z})\); rewrite the system as \(xA + yB + zC = D\). Taking dot products gives

- \((B \times C)\): \(x^1 \bar{A} + y^1 \bar{B} + z^1 \bar{C} = 0\) \(\Rightarrow x^1 = \frac{[DC]}{[ABC]}\)
- \((C \times A)\): \(B + y \bar{B} + z \bar{C} = 0\) \(\Rightarrow y = \frac{[AC]}{[ABC]}\)
- \((A \times B)\): \(D + x \bar{D} + z \bar{D} = 0\) \(\Rightarrow z = \frac{[AB]}{[ABC]}\)

**Uniqueness:** if \(xA + yB + zC = D = \bar{x}A + \bar{y}B + \bar{z}C\), then

\[
\begin{align*}
(x-x')A + (y-y')B + (z-z')C &= 0 \\
+ \xi A \times B \times C \quad \text{L.H.S. independent}
\end{align*}
\]

\(\Rightarrow (x-x', y-y', z-z') = (0, 0, 0) \Rightarrow \bar{x} = x, \bar{y} = y, \bar{z} = z\).