Lecture 43: Conic Sections

These arise in many contexts:

- Plane sections of the cone $x^2 + y^2 = z^2$
- Definition via focal points
- Definition via “directrix” (see below)
- Solution sets to quadratic polynomials in $(x, y)$
- Trajectories of particles in gravitational field (Ch. 14)

**Definition:** Given $e \in \mathbb{R}^+$, $F \in \mathbb{V}_2$, $L \subset \mathbb{V}_2$ not containing $F$, let $C_{e,F,L} = \{ X \in \mathbb{V}_2 \mid \|X - F\| = e \cdot d(X,L) \}$. This is called an ellipse if $e<1$, a parabola if $e=1$, and a hyperbola if $e>1$.

Note that $d(X,L) = |(X-P) \cdot \hat{N}|$, for any $P \in L$ and $\hat{N}$ unit vector normal to $L$.

For convenience, we choose $\hat{N}$ so that $\hat{N} \cdot (F-P) < 0$, and $P$ to minimize $d(F,P)$. Namely, Cauchy-Schwarz implies

$$\|F-P\| \geq |\hat{N} \cdot (F-P)| = d \quad \text{(indep. of $P$)}$$

has equality iff $F-P \parallel \hat{N} \Rightarrow P = F + d\hat{N}$.

The sign of $(X-P) \cdot \hat{N}$ then decides which side of $L$ (or “branch of $C$”) we are on.

With these choices, the equation becomes

$$\|X-F\| = e \left| (X-F) \cdot \hat{N} - d \right| \quad \text{ (why?)}$$
Polar form.

Choose coordinate system so that $F = 0$, $L = \{x = d\}$, $N = (1, 0)$, and write $X = (x, y) = (r \cos \theta, -r \sin \theta)$. \((\ast)\) becomes

$$||X|| = e ||X - \hat{N} - d||$$

i.e. 

$$r = e ||r \cos \theta - d||.$$ 

left branch: $(X - (d, 0)) \cdot \hat{N} < 0 \iff r \cos \theta < d$

$$r = ed - er \cos \theta \iff r = \frac{ed}{e \cos \theta + 1}$$

right branch: $r \cos \theta < d$

$$r = er \cos \theta - ed \iff r = \frac{ed}{e \cos \theta - 1}$$

Since $r > 0$, the equation forms $c > 1$. So there is only a "right branch" for hyperbolas.

Cartesian form for parabolas. \((e = 1)\)

Choosing coordinates so that $F = (0, c)$ and $L = \{y = -c\}$,

$$(x - 0)^2 \leq d(x, L)$$

$$(x - 0)^2 \leq (y + c)^2$$

$$(x, y + c) \cdot (x, y + c) = (y + c)^2$$

$$x^2 + (y + c)^2 = (y + c)^2$$

$$x^2 = 4cy$$

$$y = \frac{x^2}{4c}.$$ 

Yep, that's a parabola.
Cartesian form for ellipses & hyperbolas. \( (c < 1) \)

We don't want \( F = 0 \) here: would rather have symmetry about the origin. Equation (\( \star \)) expands as

\[
\|x - F\|^2 = \|x \cdot \hat{N} - F \cdot \hat{N} - d\|^2 = \|e \cdot \hat{N} - a\|^2 = a^2 + e^2 a^2 - 2e a x \cdot \hat{N} + x^2
\]

(\( \star \)

Want \(-x\) to also satisfy this whereas \( x \) doesn't:

\[
\|x\|^2 + 2x \cdot F + \|F\|^2 = e^2 (x \cdot \hat{N})^2 + 2e a x \cdot \hat{N} + a^2
\]

\[ x \cdot F = e a x \cdot \hat{N} \quad (\forall x) \implies F = e a \hat{N} \]

\[ F \cdot \hat{N} = e a \]

(\( \ddagger \))

\[ a = e d + e^2 a \implies e + 1 \quad \text{and} \quad \begin{cases} a = \frac{ed}{1-e^2} \\ F = \frac{e^2 d}{1-e^2} \hat{N} \end{cases} \]

(\( \star \)

In this scenario, define \(-F\) as the second focal point.

Remark: It may appear that we have imposed conditions that give special conics. In fact, this isn't really the case. Given an arbitrary \( c < 1 \) conic, replacing \( x \) by \( x' = x - T \) effects a translation of \( E, L, d, F \) and doesn't affect \( \hat{N} \). In \( (\star) \) this yields:

\[
\|x' - F\|^2 = \|e (x' - T - F) \cdot \hat{N} - d\|
\]

\[
\|x' + T - F\|^2 = \|e (x' + T - F) \cdot \hat{N} - d\|
\]

where \( F' = F - T \).

We want to have \( F' = e a \hat{N} \); to get this, simply choose \( T = F - ea \hat{N} \).

To simplify \( (\ddagger) \) further, "rotate" coordinates so that \( \hat{N} = (1, 0) \).

\[ (\Rightarrow) F = (\pm ea, 0) \implies x = \frac{\pm ea}{e} \implies \frac{d + ea}{e} = a/e \implies x = \frac{d + ea}{e} 
\]

(\( \Rightarrow \))
\[(\text{x}) \text{ becomes } x^2 + y^2 + e^2 a^2 = e^2 x^2 + a^2 \]
\[x^2 (1 - e^2) + y^2 = a^2 (1 - e^2) \]
\[\frac{x^2}{a^2} + \frac{y^2}{a^2 (1 - e^2)} = 1\]

\[e < 1: \text{ ellipse with axis lengths } \frac{a}{1 - e^2} \text{ and } \frac{b}{1 - e^2} \]

\[e > 1: \text{ hyperbola with asymptotes } y = \pm \sqrt{e^2 - 1} x \]

(Remark: Taking \(e \to 0\) we obtain a circle.)

Hyperbolas can be characterized as the set of points whose difference of its distances to the focal points remains constant. Let's do this for the "left" branch of a hyperbola symmetric about 0 with foci \(\pm F\). We need to show that \(\|X - F\| - \|X + F\|\) is constant. In fact this is

\[= |eX \cdot \hat{N} - a| - |eX \cdot \hat{N} + a| = |e| \left\{ |(X - P - d\hat{N}) \cdot \hat{N}| - |(X + P + d\hat{N}) \cdot \hat{N}| \right\} \]

(\text{always negative, } \text{always negative})

\[= -(eX \cdot \hat{N} - a) + (eX \cdot \hat{N} + a) = 2a, \text{ which is constant.} \]