Lecture 43: Conic Sections

These arise in many contexts:

- Plane sections of the cone $x^2 + y^2 = z^2$
- Definition via focal points (see HW)
- Definition via "directrix" (see below)
- Solution sets to quadratic polynomials in $(x,y)$
- Trajectories of particle in gravitational field (Ch. 14)

**Definition:** Given $e \in \mathbb{R}^+$, $F \in \mathbb{R}^2$, $L \subseteq \mathbb{R}^2$ line not containing $F$, let $C_{e,F,L} := \{ X \in \mathbb{R}^2 \mid \| X - F \| = e \cdot d(X,L) \}$. This is called an **ellipse** if $e < 1$, a **parabola** if $e = 1$, and a **hyperbola** if $e > 1$.

Note that $d(X,L) = |(X-P) \cdot \hat{N}|$, for any $P \in L$ and $\hat{N}$ unit vector normal to $L$.

For convenience we choose $\hat{N}$ so that $\hat{N} \cdot (F-P) < 0$, and $P$ to minimize $d(F,P)$. Namely, Cauchy-Schwarz $\Rightarrow$ $\| F - P \| \geq | \hat{N} \cdot (F-P) | = d$ (indep. of $P$) holds equiv iff $F-P \parallel \hat{N}$ $\Rightarrow$ $P = F + d\hat{N}$.

The sign of $(X-P) \cdot \hat{N}$ then decides which side of $L$ (or "branch of $C$") we are on.

With these choices, the equation becomes

\[ \| X - F \| = e \left| (X - F) \cdot \hat{N} - d \right| . \] (why?)
Polar form.

Choose coordinate system so that \( F = 0 \), \( L = \{x = d\} \), \( N = (1, 0) \), and write \( X = (x, y) = (r \cos \Theta, r \sin \Theta) \). (2) becomes

\[
||X|| = e \left| X - \hat{N} - d \right|
\]

i.e. \( r = e \left| r \cos \Theta - d \right| \).

**left branch:** \((X - (d, 0)).\hat{N} < 0 \iff r \cos \Theta < d \)

\[
r = ed - er \cos \Theta \Rightarrow r = \frac{ed}{e \cos \Theta + 1}
\]

**right branch:** \(r \cos \Theta > d \)

\[
r = er \cos \Theta - ed \Rightarrow r = \frac{ed}{e \cos \Theta - 1}
\]

Since \( r > 0 \), the equation for \( e > 1 \) : so there is only a “right branch” for hyperbolas.

Cartesian form for parabolas. \((e = 0)\)

Choosing coordinates so that \( F = (0, c) \) and \( L = \{y = -c\} \),

\[
(1) \quad (X - F) \cdot (X - F) = d(X, L)
\]

\[
(x - p)^2 = (y + c)^2
\]

\[
(x, y - c) \cdot (x, y - c) = (y + c)^2
\]

\[
x^2 + (y - c)^2 = (y + c)^2
\]

\[
x^2 = 4cy
\]

\[
y = x^2/4c
\]

Yep, that’s a parabola.
Cartesian form for ellipses & hyperbolas \((e \neq 1)\)

We don't want \(F = 0\) here; would rather have symmetry about the origin. Equation (4) expands as

\[
\|x-F\|^2 = e \left( x \cdot \hat{N} - F \cdot \hat{N} - d \right) = e (x \cdot \hat{N} - a)^2,
\]

(5)

\[
\|x\|^2 - 2x \cdot F + \|F\|^2 = e^2 (x \cdot \hat{N})^2 - 2ea x \cdot \hat{N} + a^2
\]

Want \(-x\) to also satisfy this whereas \(x\) does:

\[
\|x\|^2 + 2x \cdot F + \|F\|^2 = e^2 (x \cdot \hat{N})^2 + 2ea x \cdot \hat{N} + a^2
\]

\[
\Rightarrow x \cdot F = ea x \cdot \hat{N} \quad \forall x \quad \Rightarrow \quad F = ea \hat{N}
\]

\[
\Rightarrow \quad F \cdot \hat{N} = ea \quad (4) \quad \Rightarrow \quad a = ed + e^2a \quad \Rightarrow \quad e \neq 1 \quad \text{and} \quad \begin{cases} a = \frac{ed}{1-e^2} \\ F = \frac{e^2d}{1-e^2} \hat{N} \quad (= ea \hat{N}) \end{cases}
\]

\[
\Rightarrow \quad \text{eqn. is} \quad x \cdot x + e^2a^2 = e^2 (x \cdot \hat{N})^2 + a^2. \quad (5x)
\]

In this scenario, define \(-F\) as the second focal point.

Remark: It may appear that we have imposed conditions that give special conics. In fact, this isn't really the case. Given an arbitrary \(e \neq 1\) conic, replacing \(x\) by \(x' = x-T\) offsets a translation \(\hat{C}, L, \hat{d}, F\) and doesn't offset \(\hat{N}\). In (5x), this yields

\[
\|x' + T - F\| = e \left| (x' + T - F) \cdot \hat{N} - d \right|
\]

\[
\|x' - F\| = e \left| (x' - F') \cdot \hat{N} - d \right| \quad \text{where} \quad F' = F - T.
\]

We want to have \(F' = ea \hat{N}\); to get this, simply choose \(T = F - ea \hat{N}\).

To simplify (5x) further, "rotate" coordinates so that \(N = (1, 0)\).

\[
(\Rightarrow) \quad \pm F = (\pm ea, 0) \quad \Rightarrow \quad \lambda = \frac{(1-e^2)e}{a} = \frac{a^2}{e} - ea
\]

\[
d + ea = a/e \quad \Rightarrow \quad x = \frac{d}{e} \quad \text{is the} \quad \text{director.} \quad \)
$(\text{x, y})$ becomes \[ x^2 + y^2 + e^2 a^2 = e^2 x^2 + a^2 \]
\[ x^2(1-e^2) + y^2 = a^2(1-e^2) \]
\[ \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1 \]

$e < 1$: ellipse with axis lengths $a, \frac{a}{e}$

$e > 1$: hyperbola with asymptotes $y = \pm \sqrt{e^2 - 1} x$

Remark: taking $e \to 0$ we obtain a circle.

Hyperbolas can be characterized as the set of points whose difference of its distances to the focal points remains constant. Let's do this for the "left" branch of a hyperbola, symmetric about 0 with focus $F$. We need to show that $||x-F|| - ||x+F||$ is constant. In fact this is

\[ = \left| e \langle x, \hat{N} \rangle - a \right| - \left| e \langle x, \hat{N} \rangle + a \right| = |e| \left| \left( \langle x - P, \hat{N} \rangle \right) \hat{N} - \left( \langle x + F, \hat{N} \rangle \right) \hat{N} \right| \]

always negative (left branch) always negative

\[ = - (e \langle x, \hat{N} \rangle - a) + (e \langle x, \hat{N} \rangle + a) \]

= $2a$, which is constant.