Lecture 45: Arc-length

Let's begin with a bit of review of last time. The notation in Apostol for vector-valued functions keeps "evolving"—first $F(x)$, then $X(t)$, now $\vec{r}(t)$ (my "-" = his boldface) when discussing curvilinear motion. I encourage the use of the letters:

- $\vec{r}(t) =$ position at time $t$
- $\vec{v}(t) =$ velocity at time $t$, $v(t) := \|\vec{v}(t)\| =$ speed
- $\vec{a}(t) := \vec{v}'(t) = \vec{v}(t)$ = acceleration.

Example 1: $\vec{F}(t) = (\alpha \cosh \omega t, \beta \sinh \omega t)$

$$\vec{a}(t) = \vec{v}''(t) = \omega^2 \vec{v}(t)$$

is in the same direction as $\vec{v}$

- opposite to the scenario for elliptic motion $\vec{r}(t) = (\alpha \cos \omega t, \beta \sin \omega t)$.

Example 2: $\vec{r}(t) = (3 \cos t, 3 \sin t, 4t)$ traces out a helix. Its tangent line at $t = \pi/2$ is $L \left( \vec{r}(\pi/2), \vec{v}(\pi/2) \right) = L \left( (0, 3, 2\pi), (-3, 0, 4) \right)$ which is parameterized by $\vec{q}(s) = (-3s, 3, 2\pi + 4s)$.

Problem: The velocity of a bug at time $t$ is given by $\vec{v}(t) = (t, t^3, t^3)$. It starts flying from the point $(1,1,1)$ at $t=0$. Find (a) its position at time $t=1$ and (b) the tangent line (at $t=1$) to the curve $C$ it traces out.
Definition: (i) When $F'(t) \neq 0$, the unit tangent vector $T(t) = \frac{F'(t)}{||F'(t)||}$ is defined. (ii) When $T'(t) \neq 0$, the unit normal vector $N(t) = \frac{T'(t)}{||T'(t)||}$ is defined. (iii) If $T, N$ are defined at $t_0$, the osculating plane to $C$ at $r(t_0)$ is given by $M(T(t_0), \{T(t_0), N(t_0)\})$.

Example 3: In Ex 2 $\mathbf{r} = (-3 \sin t, 3 \cos t, t) \implies ||\mathbf{r}'|| = 5$ (constant)

$\implies T = \frac{\mathbf{r}'}{||\mathbf{r}'||} = (\frac{3}{5} \sin t, \frac{3}{5} \cos t, 1), T' = \frac{3}{5} (\cos t, \sin t, 0)$

$\implies N = \frac{T'}{||T'||} = (-\cos t, -\sin t, 0)

Taking $t_0 = \frac{\pi}{4}, T(t_0) = (\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}), N(t_0) = (0, -1, 0)$, so the normal vector to $M$ is

$B = T(t_0) \times N(t_0) = (\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2})$. Hence the equation of $M$ is $\mathbf{r} - \mathbf{r}(t_0) \cdot B = 0$ i.e. $\frac{4}{5} x + \frac{3}{5} (t - 2\pi) = 0$.

Note that since $T = \frac{\mathbf{r}'}{||\mathbf{r}'||} = \frac{\mathbf{v}}{||\mathbf{v}||}$, $\mathbf{v} = vT$.

Differentiating gives $\ddot{\mathbf{r}} = v' T + vT' = \frac{v'}{v} T + \frac{v}{v} ||T'|| N$, so that $v'$ and $\ddot{\mathbf{r}}$ lie in the osculating plane.

Now let $F : [a,b] \rightarrow V_n$, trace out $C = \mathbf{r}([a,b])$, and $P = \{t_0, t_1, \ldots, t_m\}$ (a=\(t_0, c_1, \ldots, t_m=b\) be a partition of [a,b]).

Let $\tau(P)$ denote the polygon with vertices $\{\mathbf{r}(t_k)\}$, with total length $|\tau(P)| = \sum_{k=1}^{m} ||\mathbf{r}(t_k) - \mathbf{r}(t_{k-1})||$.

Definition: (i) $C$ is rectifiable $\iff \mathcal{L} = \{\tau(P) \mid P \text{ partition of } [a,b]\}$ is bounded above.

(ii) If $C$ is rectifiable, define its length (L(C) or L(\([a,b]\)) by $\sup(\mathcal{L})$. 

Theorem: If \( \dot{v}(t) \) is continuous, then \( C \) is rectifiable and
\[
\ell(C) = \int_a^b \| \dot{v}(t) \| dt.
\]

Ex 4/ \( \dot{v}(t) = (2 \cos t, 2 \sin t), \quad t \in [0, \theta] \Rightarrow \) arc of a circle.

\[
\Rightarrow \dot{v}(t) = (-2 \sin t, 2 \cos t) \Rightarrow \| \dot{v}(t) \| = 2. \quad \text{So } \ell(C) = 2 \theta. \quad \Box
\]

If we let \( b \) move (take \( b = t \)), the arc length from \( \dot{v}(a) \) to \( \dot{v}(t) \) is
\[
\ell(C) = \int_a^t \| \dot{v}(s) \| ds, \quad \text{and } \ell(C) = \int_a^t \| \dot{v}(s) \| ds.
\]

A parameterization by arc length is given by \( (3 \cos \frac{\pi s}{5}, 3 \sin \frac{\pi s}{5}, \frac{4 s}{5}) \).

Ex 5/ \( \dot{v}(t) = (3 \cos t, 3 \sin t, 4 t), \quad t \in (0, \frac{\pi}{2}) \Rightarrow \| \dot{v}(t) \| = 5 \) (constant)

\[
\Rightarrow \ell(C) = \int_0^{\frac{\pi}{2}} 5 \, dt = \frac{5 \pi}{2}. \quad \text{A parameterization by arc length is given by } (3 \cos \frac{\pi s}{5}, 3 \sin \frac{\pi s}{5}, \frac{4 s}{5}). \quad \Box
\]

Ex 6/ \( \dot{v}(t) = (t, f(t)), \quad t \in [a, b], \) parameterizes the graph \( \Gamma \) of \( y = f(x) \) for \( x \in [a, b] \).

Since \( \dot{\Gamma} = (1, \dot{f}(t)) \), \( \| \dot{\Gamma} \| = \sqrt{1 + (\dot{f}(t))^2} \) and so
\[
\ell(\Gamma) = \int_a^b \sqrt{1 + (\dot{f}(t))^2} \, dt. \quad \Box
\]

Problem: Find the arc length of the curve traced out by \( \dot{v}(t) = (t, 3 \cos t, 6 t^3) \) on \([0, \pi/2]\).

Proof of Theorem: First, \( \| \dot{v}(t) \| = \sum_{k=1}^n \| \dot{v}(t_k)-\dot{v}(t_{k-1}) \| = \sum_{k=1}^n \| \int_{t_{k-1}}^{t_k} \dot{v}(t) \, dt \| \)

\[
\leq \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \| \dot{v}(t) \| \, dt = \int_a^b \| \dot{v}(t) \| \, dt \quad \text{gives an upper bound on } \ell(C), \quad \text{and } \sup(\ell) \leq \text{any upper bound. So } C \text{ is rectifiable and } \ell(C) \leq \int_a^b \| \dot{v}(t) \| \, dt. \quad \text{It remains to prove the opposite inequality.}
\]

To do this, we first verify an "additivity property": if
\( C_1 = \tilde{r}((x,1)) \) and \( C_2 = \tilde{r}((c,5)) \), then \( \lambda(C_1) = \lambda(C_2) + \lambda(C_3) \). [Proof:
\[
\Delta(a,b) = \lambda(C_3) + \lambda(C_4)
\]
if \( P_i, P_2 \) are partitions of \( [a,c] \) and \( [c,b] \), then taking \( P = P_1 \cup P_2 \) gives
\[
\lambda(C) \geq |\pi(P)| = |\pi(P_1)| + |\pi(P_2)| \Rightarrow C_i \text{ rectifiable } \Rightarrow \left| \pi(P_i) \right| \leq \lambda(C_1) - |\pi(P_1)| \Rightarrow
\]
\[
\lambda(C_1) \leq \lambda(C) - |\pi(P_2)| \Rightarrow |\pi(P_2)| \leq \lambda(C) - \lambda(C_1) \Rightarrow
\]
\[
\lambda(C_2) \leq \lambda(C) - \lambda(C_1) \Rightarrow \lambda(C_1) + \lambda(C_2) \leq \lambda(C).\] Commonly, let \( P \) be arbitrary and let \( P_c = P \cup \{c\} = P_1 \cup P_2 \).

By the triangle inequality (see figure),
\[
|\pi(P)| \leq |\pi(P_c)| = |\pi(P_1)| + |\pi(P_2)| \leq \lambda(C_1) + \lambda(C_2) \Rightarrow \lambda(C) \leq \lambda(C_1) + \lambda(C_2).\]

Now let \( S(t) := \begin{cases} 0, & t = a \\ \Lambda(a,t), & t > a \end{cases} \)
which is increasing since \( \lambda(C_1) - S(t) = \Lambda(a,t) - \Lambda(a,t_i) = \Lambda(t, t_i) \geq 0 \)
using the additivity just proved. Define also \( f(t) := \int_a^t v(u) \, du \), which has \( f'(t) = v(t) \) \( [v \text{ cts. } \Rightarrow v \text{ cts.}] \). Then by the key part of proof,
\[
\left\| \frac{\tilde{r}(t+h) - \tilde{r}(t)}{h} \right\| \leq \Lambda(t, t+h) = S(t+h) - S(t) \leq \int_t^{t+h} v(u) \, du. \]
Hence
\[
\left\| \frac{\tilde{r}(t+h) - \tilde{r}(t)}{h} \right\| \leq \frac{S(t+h) - S(t)}{h} \leq \frac{1}{h} \int_t^{t+h} v(u) \, du = \frac{f(t+h) - f(t)}{h},
\]
the end terms of which limit to \( \| \tilde{r}'(t) \| = v(t) = f'(t) \). By the Square Theorem, we deduce the middle term's limit \( S(t) \) is also \( v(t) \). \( \square \)