Curvilinear motion in polar coordinates

Let \( \mathbf{r} : I \to \mathbb{V}^2 \) be a vector-valued function (with continuous \( 1^{\text{st}} \) & \( 2^{\text{nd}} \) derivatives). Write \( \dot{\mathbf{r}}(t) = r(t)(\cos \theta(t), \sin \theta(t)) = r(t) \Theta'(t) \), or \( \mathbf{r}' = r \Theta' \) for short. We have \( \frac{d}{dt} \theta(t) = \frac{d}{dt} (\cos \theta(t), \sin \theta(t)) = \theta'(t)(-\sin \theta(t), \cos \theta(t)) = \theta'(t)(\cos(\theta(t) + \frac{\pi}{2}), \sin(\theta(t) + \frac{\pi}{2})) = \theta'(t) \Theta'(t + \frac{\pi}{2}) \), or \( \dot{\theta} = \Theta'(t + \frac{\pi}{2}) \) for short. So then we have

- \( \mathbf{v} = (r \Theta')' = r' \Theta + r \Theta' \Theta + \frac{\pi}{2} \)
- \( \mathbf{a} = \mathbf{v}' = r'' \Theta + r' \Theta' \Theta + \frac{\pi}{2} + r \Theta'' \Theta + \frac{\pi}{2} + \frac{\pi}{2} \Theta' \Theta' \Theta + \frac{\pi}{2} \Theta' \)
- \( \mathbf{v} = \| \mathbf{v} \| = \sqrt{(r')^2 + r^2 (\Theta')^2} \) since \( \Theta, \Theta + \frac{\pi}{2} \) are orthogonal unit vectors which can be used to calculate curvature as well.

Proof of Kepler's 1st Law

Recall where we left off in our proof of the 2nd law:

- \( \mathbf{F} = m \mathbf{a} \) & \( \mathbf{F} = -\frac{GMm}{r^2} \) \( \Rightarrow \mathbf{a} = -\frac{GM}{r^2} \mathbf{r} = -\frac{GM}{r^2} \Theta \)
- \( \mathbf{r} \times \mathbf{a} = 0 \Rightarrow \mathbf{r} \times \mathbf{v} = \mathbf{C} = C \mathbf{k} \) (constant)

Now consider

\( \mathbf{a} \times \mathbf{C} = \left(-\frac{GM}{r^2} \Theta \right) \times (r^2 \Theta' \mathbf{k}) = GM \Theta' \Theta + \frac{\pi}{2} \).

See Apostol for proof of 3rd law (easy but technical, not conceptual).
Integrating both sides gives \( \ddot{r} \times \ddot{c} = GM \tau_0 + \ddot{c} = GM (\tau_0 + \ddot{c}) \), and dotting with \( \ddot{r} \) gives

\[
GM (\tau_0 + \ddot{c}) \cdot \ddot{r} = \ddot{r} \cdot (\ddot{r} \times \ddot{c}) = \ddot{r} \cdot (\ddot{r} \times \ddot{r}) = \ddot{c} \cdot \ddot{c} = C^2.
\]

Writing \( \phi \) for the angle between \( \ddot{r}(t) \) and \( \ddot{c} \), and \( e := \| \ddot{c} \| \),

\[
GM r (1 + e \cos \phi) = C^2 \quad \text{becomes} \quad \text{(with } d := \frac{C^2}{GM e})
\]

\[r(t) = \frac{ed}{1 + e \cos \phi}\]

which is the polar equation of a conic of eccentricity \( e \) and focus at the origin \( O \).

Planets are planets because they are in orbit (and not visitors from deep space which are flung back therefrom), and so this conic must be an ellipse, i.e. \( e \in (0, 1) \).

\[\text{For the remainder of the course we will be studying (real) vector spaces, or linear spaces in Apostol's terminology.}\]

**Definition:** A (real) vector space is a set \( V \) together with two binary operations

\[ + : V \times V \to V \quad \text{and} \quad \cdot : \mathbb{R} \times V \to V \]

(vector addition \( (x, y) \to x + y \))

(scalar mult. \( (r, x) \to r x) \)

and an "zero element" \( 0 \in V \) such that (for all \( r, s, x, y \))

- \( x + y = y + x \)
- \( (x + y) + z = x + (y + z) \)
- \( r(sx) = (rs)x \)
- \( (r + s)x = rx + sx \)
- \( 0 + X = X \)
- \( 1X = X \)
- \( X + (-1)X = 0 \)
- \( r(x + y) = rx + ry \)

herefrom and "\(-X\)"
Remark: $0$ is unique: if $0'$ also satisfies this property, then
$0' = 0 + 0' = 0' + 0 = 0$.

$-x$ is unique: if $x + y = 0$, then adding $-x$ to both sides gives
$-x + (x + y) = -x + 0 \Rightarrow (-x + x) + y = -x \Rightarrow y = -x$.

$0 \times 0 = 0: \quad 0 \times + 0 \times = (0 + 0) \times = 0 \times$
now add $-0 \times$ to both sides.

Ex 1 / Obvious example: $V_n$
Along the linear span $L(S)$ of a subset $S \subset V_n$.
Or, the subset $W \subset V_n$ of vectors perpendicular to $S$.

Ex 2 / Less obvious: we can take $V$ to be a set of functions:
- all real-valued functions on $[a,b]$.
- all real-valued functions on $[a,b]$ with $f(a) = 0$ (why doesn’t this work?)
- all continuous real-valued funs. on $[a,b]$ with $f(a) = 1$ work?
- all differentiable real-valued funs. on $[a,b]$.
- polynomials with real coefficients.
- polynomials of degree $\leq d$ with real coefficients.
  (Why don’t polynomials of degree $d$ work?)
- solutions of $y'' + ay' + by = 0$ (why not $F(x)$?)

Considering "vector spaces of functions" is not just the right way to study higher-order ODEs: It is how you break a sound wave or electrical signal into its constituent frequencies (Fourier analysis).

Problem: (1) Let $V := \mathbb{R} \times \mathbb{R}$, with operations $(x, y) + (a, b) = (x + a, y + b)$
          $c(x, y) = (cx, cy)$.
Is this a vector space?

(2) What if we replace the operations by
      $(x, y) + (a, b) = (x + a, 0)$ and $c(x, y) = (cx, 0)$?