Most of today was finishing up Lecture 50, but I began with the following example which used a result from your HW:

that \( g_0(x) = \frac{1}{\sqrt{\pi}} \) and \( \{ g_n(x) = \sqrt{\frac{2}{\pi}} \cos(nx) \}_{n=1}^{\infty} \) constitute an orthonormal set in the vector space \( C(0, \pi) \) with inner-product \( (f, g) = \int_{0}^{\pi} f(x)g(x) \, dx \). We shall assume that their "Span", ignoring issues of convergence, contains all nice enough functions.

Consider the function \( f(x) = (x-\pi)^2 \in C(0, \pi) \):

\[
\begin{align*}
\text{Since the } \{g_n\} \text{ are o.n. and } f \text{ is in their span, then pretending infinite sums are like finite ones, we can write} \\
f &= \sum_{n=0}^{\infty} (f, g_n) g_n \\
&= (f, \frac{1}{\sqrt{\pi}}) \frac{1}{\sqrt{\pi}} + \sum_{n=1}^{\infty} (f, \sqrt{\frac{2}{\pi}} \cos(nx)) \sqrt{\frac{2}{\pi}} \cos(nx) \\
&= \frac{1}{\pi} (f, 1) + \frac{2}{\pi} \sum_{n=1}^{\infty} (f, \cos(nx)) \cos(nx).
\end{align*}
\]
Now \((f, 1) = \int_0^\pi (x-\pi)^2 \, dx = \left(\frac{\pi - x}{3}\right) \bigg|_0^\pi = \frac{\pi^3}{3}\), while
\[
(f, \cos(nx)) = \int_0^\pi (x-\pi)^2 \cos(nx) \, dx = \left(\frac{\pi - x}{3}\right) \left(-\frac{1}{n} \sin(nx) \bigg|_0^\pi - \frac{2}{n} \int_0^\pi (x-\pi) \sin(nx) \, dx\right)
\]
\[
h = (x-\pi)^2 \quad dv = \cos(nx) \, dx
\]
\[
du = 2(x-\pi) \, dx \quad v = \frac{1}{n} \sin(nx)
\]
\[
= -\frac{2}{n} \int_0^\pi (x-\pi) \sin(nx) \, dx = -\frac{2}{n} \left(\frac{1}{n} (x-\pi) \cos(nx) \bigg|_0^\pi + \frac{1}{n} \int_0^\pi \cos(nx) \, dx\right)
\]
\[
h = x-\pi \quad dv = \sin(nx) \, dx
\]
\[
du = dx \quad v = \frac{1}{n} \cos(nx)
\]
\[
= \frac{2\pi^2}{n^2}.
\]

So
\[
f(x) = \frac{1}{\pi} \left(\frac{\pi^3}{3} + \frac{2}{\pi} \sum_{n=1}^\infty \frac{4}{n^2} \cos(nx)\right) = \frac{\pi^2}{3} + \sum_{n=1}^\infty \frac{4}{n^2} \cos(nx)
\]

\[
(x-\pi) = 2(x-\pi)^2 \quad \text{for } x \in [0, \pi].
\]

Evaluating this at \(x = 0\), we get
\[
\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^\infty \frac{4}{n^2}
\]
\[
\frac{2\pi^2}{3} = 4 \sum_{n=1}^\infty \frac{1}{n^2}
\]

Thus
\[
\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}.
\]