Lecture 52: Isomorphisms & Inverses

Let \( T : V \rightarrow W \) be a linear transformation.

[Note: Most of this will be re-done next semester in terms of matrices.]

Definition: (i) \( T \) is onto or surjective if \( \text{im}(T) = W \),
    in which case we write \( T : V \rightarrow W \)

(ii) \( T \) is 1-to-1 or injective if \( \vec{v} \neq \vec{v}' \Rightarrow T\vec{v} \neq T\vec{v}' \)
    (i.e. \( T\vec{v} = T\vec{v}' \Rightarrow \vec{v} = \vec{v}' \))

In this case we write \( T : V \rightarrow W \).

(iii) If \( T \) is 1-to-1 & onto, it is an isomorphism,
    which is written \( T : V \cong W \). Two vector spaces are called isomorphic \( (V \cong W) \) if there exists an isomorphism between them, in either direction (though we'll soon see the two are equivalent).

Proposition 2: \( T \) is 1-to-1 \( \iff \ker(T) = \{0\} \).

Proof: \((\Rightarrow)\) is clear: only \( \vec{0} \) can go to \( \vec{0} \)

\((\Leftarrow)\) Suppose \( \ker(T) = \{0\} \), and let \( T\vec{v} = T\vec{v}' \).
    By linearity, \( \vec{0} = T\vec{v} - T\vec{v}' = T(\vec{v} - \vec{v}') \), so
    \( \vec{v} - \vec{v}' \in \ker(T) = \{0\} \Rightarrow \vec{v} - \vec{v}' = \vec{0} \Rightarrow \vec{v} = \vec{v}' \). \( \square \)

Ex: \( W \leq V \) subspace of inner-product space \( V \), inclusion \( W \rightarrow V \), projection \( V \rightarrow W \). //
Proposition 2: A linear transformation $T: V \to W$ is determined by where it sends a basis. That is, if $\{\vec{v}_1, \ldots, \vec{v}_n\} \subseteq V$ is a basis, and $\vec{w}_1, \ldots, \vec{w}_n \in W$ (not necessarily distinct!), then there is exactly one $T$ with $T\vec{v}_k = \vec{w}_k$ for $k = 1, \ldots, n$.

If $\{\vec{w}_1, \ldots, \vec{w}_n\}$ are independent, then $T$ is 1-to-1.

If they span $W$, then $T$ is onto. If they're a basis, $T$ is an isomorphism.

Proof: Any element $\vec{v} = \sum_{k=1}^n c_k \vec{v}_k \in V$ would have to be sent to $(\vec{v}) = \sum_{k=1}^n c_k T\vec{v}_k$ by linearity; and this also gives a definition of $T$.

If $\{\vec{w}_1, \ldots, \vec{w}_n\}$ is L.I., then $\vec{v} \in \ker(T)$ implies

$$0 = T\vec{v} = T\left(\sum_{k=1}^n c_k \vec{v}_k\right) = \sum_{k=1}^n c_k T\vec{v}_k = \vec{0} \quad (c_k = 0)$$

$\Rightarrow \vec{v} = 0$, so $T$ is injective.

If any vector $\vec{w} \in W$ is in the linear span of $\vec{w}_1 = T\vec{v}_1, \ldots, \vec{w}_n = T\vec{v}_n$, then $\vec{w} = \sum_{k=1}^n c_k T\vec{v}_k = T(\sum_{k=1}^n c_k \vec{v}_k) \in \text{im}(T)$.

\[\square\]

Example: Let $T: \mathbb{V}_3 \to \mathbb{P}_2$ be the L.T. determined by $T(\hat{i}) = 1$, $T(\hat{j}) = x$, $T(\hat{k}) = x^2$. By the Prop., this is an isomorphism.

//

Proposition 3: Assume $V$, $W$ are finite-dimensional. Then

$$\dim V = \dim W \iff V \cong W.$$  

Proof: ($\Leftarrow$) Suppose $T: V \to W$ (or vice-versa) is an $\cong$.

Then ker$(T) = \{0\} \Rightarrow \text{mult}(T) = 0 \Rightarrow \dim(V) = \text{rank}(T) + \dim(\text{ker}(T)) = \dim(\text{im}(T)) = \dim(W).$
Suppose \( \dim V = n = \dim W \). Let \( \{ \bar{v}_1, \ldots, \bar{v}_n \} \) and \( \{ \bar{w}_1, \ldots, \bar{w}_n \} \) be bases. The L.T. defined (via Prop 2) to send \( \bar{v}_j \mapsto \bar{w}_j \) is an isomorphism.

We may compose linear transformations (just as we would compose any functions): e.g.,

\[
U \xrightarrow{R} V \xrightarrow{T} W
\]

is written \( TR \) (or \( TR \)). This is linear because

\[
TR(\alpha \bar{v} + \beta \bar{w}) = T(R(\alpha \bar{v} + \beta \bar{w})) = T(\alpha R \bar{v} + \beta R \bar{w}) = \alpha TR \bar{v} + \beta TR \bar{w}.
\]

**Definition:** Given \( T: V \to W \), \( S: W \to V \) is

(i) a right inverse for \( T \) if \( T \circ S = I_W \)

(ii) a left inverse for \( T \) if \( S \circ T = I_V \)

(iii) an inverse of \( T \) if it is both: write \( S = T^{-1} \).

(In this case, we say \( T \) is invertible.)

**Theorem:** \( T: V \to W \) has...

(i) a right inverse \( \iff \) \( T \) is onto

(ii) a left inverse \( \iff \) \( T \) is 1-to-1

(iii) an inverse \( \iff \) \( T \) is an isomorphism.

**Proof:** (i) \( (\Rightarrow) \): If \( TS = I_W \), then any \( \bar{w} = I_w(\bar{w}) = T(S(\bar{w})) \in \text{im}(T) \).

\( (\Leftarrow) \): I prove this only in the finite dimensional case (otherwise it requires non-dimensional bases & the Axiom of Choice).

Let \( \{ \bar{v}_1, \ldots, \bar{v}_m \} \) = basis of \( W \) and choose \( \bar{v}_1, \ldots, \bar{v}_m \) with \( T\bar{v}_j = \bar{w}_j \) (since \( T \) is onto). Define \( S \) to and \( \bar{v}_j \mapsto \bar{w}_j \).
(i) \( \Rightarrow \): If \( ST = I \), then \( T\hat{v} = \hat{v} \Rightarrow \hat{v} = I_{n} \hat{v} = S(T\hat{v}) = S(\hat{v}) = \hat{v} \).

(\Leftarrow): Some comment as above \( \Rightarrow \ker(T) = \{0\}. \)

Let \( \{\hat{v}_{1}, \ldots, \hat{v}_{n}\} = \text{basis of } \text{im}(T). \) (So \( \hat{v}_{j} = T\hat{v}_{j} \) for some \( \{\hat{v}_{1}, \ldots, \hat{v}_{n}\} \subseteq V. \) ) Extend this to a basis \( \{\hat{w}_{1}, \ldots, \hat{w}_{n}, \hat{w}_{n+1}, \ldots, \hat{w}_{m}\} \) of \( W. \) Define \( S \) to send \( \hat{w}_{i} \rightarrow \hat{v}_{i}, \ldots, \hat{w}_{n} \rightarrow \hat{v}_{n}; \hat{w}_{n+1} \rightarrow 0. \) Then \( \hat{w}_{m} \rightarrow 0. \)

(ii) \( \Leftarrow \): Any \( \hat{w} \in W \) is \( T \) of some \( \hat{v}_{m} \in V, \) since \( T \) is onto. Since \( T \) is 1-to-1, this \( \hat{v}_{m} \) is unique. So define \( S\hat{w} := \hat{v}_{m} \) for each \( \hat{w}. \) Clearly \( T(S\hat{w}) = T(\hat{v}_{m}) = \hat{w} \) by definition, and clearly \( S(T\hat{v}) = \hat{v} \) as well. So \( S = T^{-1}. \) \( \square \)

Ex. Let \( a_{0}, a_{1}, \ldots, a_{n} \in \mathbb{R} \) be distinct, and consider the evaluation map \( T : P_{n} \rightarrow \mathbb{R}_{n+1}, \)

\[ f(t) \mapsto (f(a_{0}), \ldots, f(a_{n})). \]

Is this invertible / an isomorphism?

Set \( f_{i}(t) := \prod_{k=0}^{n} \frac{t-a_{k}}{a_{i}-a_{k}} \) \( (\text{product of these as } k \text{ runs from } 0 \text{ to } n \text{ skipping } i) \).

Then \( f_{i}(a_{j}) = \prod_{k=0}^{n} \frac{a_{j}-a_{k}}{a_{i}-a_{k}} \) \( = \{0 \text{ if } j \neq i \} \) \( = \{1 \text{ if } j = i \} \).

Define \( S(b_{0}, \ldots, b_{n}) := \sum_{i=0}^{n} b_{i} f_{i}(t), \) we have \( TS(b_{0}, \ldots, b_{n}) = T(\sum_{i=0}^{n} b_{i} f_{i}(t)) = \sum_{i=0}^{n} b_{i} T f_{i}(t) \)

\( = (\sum b_{i} f_{i}(a_{0}), \sum b_{i} f_{i}(a_{1}), \ldots, \sum b_{i} f_{i}(a_{n})) = (b_{0}, b_{1}, \ldots, b_{n}). \)
So \( T \circ S = \text{Id}_{V_{n+1}} \Rightarrow S \) is a right inverse \( \Rightarrow T \) is onto. By Rank + Nullity,
\[
\dim(V_{n+1}) = \dim(\text{im}(T)) + \dim(\ker(T)) = n + 1
\]
\[
\Rightarrow \ker(T) = \{0\} \Rightarrow T \text{ is } 1 \text{-to-1} \Rightarrow T \text{ is an isomorphism.}
\]

Notice that what we've really done here is shown that the function \( \sum_{2=0}^{n} f(x) \) takes prescribed values \( b_0, \ldots, b_n \) at \( x_0, \ldots, x_n \). This is called Lagrange interpolation. //

We conclude with the following observation, which came up in the example just done:

**Proposition 4:** If \( V \) \& \( W \) are of the same finite dimension \( n \), then the following are equivalent for \( T : V \to W \):

(a) \( T \) is onto
(b) \( T \) is 1-to-1
(c) \( T \) is an isomorphism

**Proof:** Use Rank + Nullity:

\[
\dim(\text{im}(T)) + \dim(\ker(T)) = n.
\]
If (a) holds, \( \dim(\text{im}(T)) = \dim(W) = n \Rightarrow \ker = \{0\} \Rightarrow (c) \).
If (b) holds, \( \dim(\ker(T)) = 0 \Rightarrow \dim(\text{im}(T)) = n \Rightarrow \dim(T) = W \).
Since (a)+(b) is equivalent to (c), done.