Lecture 52: Isomorphisms & Inverses

Let \( T : V \rightarrow W \) be a linear transformation.

**Note:** Most of this will be re-done next semester in terms of matrices.

**Definition:**
1. \( T \) is **onto** or **surjective** if \( \text{im}(T) = W \), in which case we write \( T : V \rightarrowrightarrow W \).
2. \( T \) is **1-to-1** or **injective** if \( \overline{v} \neq \overline{v}' \implies T\overline{v} \neq T\overline{v}' \).
   (i.e. \( T\overline{v} = T\overline{v}' \implies \overline{v} = \overline{v}' \))
   In this case we write \( T : V \leftarrowrightarrow W \).
3. If \( T \) is 1-to-1 \& onto, it is an **isomorphism**, which is written \( T : V \congrightarrow W \). Two vector spaces are called **isomorphic** \( \left( V \cong W \right) \) if there exists an isomorphism between them, in either direction (though we'll soon see the two are equivalent).

**Proposition 1:** \( T \) is 1-to-1 \( \iff \) \( \text{ker}(T) = \{ \overline{0} \} \).

**Proof:**

\((\Rightarrow)\) is clear: only \( \overline{0} \) can go to \( \overline{0} \).

\((\Leftarrow)\) Suppose \( \text{ker}(T) = \{ \overline{0} \} \), and let \( T\overline{v} = T\overline{v}' \).

By linearity, \( \overline{0} = T\overline{v} - T\overline{v}' = T(\overline{v} - \overline{v}') \), so \( \overline{v} - \overline{v}' \in \text{ker}(T) = \{ \overline{0} \} \) \implies \( \overline{v} - \overline{v}' = \overline{0} = \overline{0} \implies \overline{v} = \overline{v}' \). \( \square \)

**Example:** \( W \leq V \) subspace of inner product space \( \implies \) inclusion \( W \leqrightarrow V \), projection \( V \rightarrowrightarrow W \). //
Proposition 2: A linear transformation $T : V \rightarrow W$ is determined by where it sends a basis. That is, if $\{\hat{v}_1, \ldots, \hat{v}_n\}$ is a basis, and $\hat{w}_1, \ldots, \hat{w}_n \in W$ (not necessarily distinct!), then there is exactly one $T$ with $T \hat{v}_k = \hat{w}_k$ for $k = 1, \ldots, n$. If $\{\hat{w}_1, \ldots, \hat{w}_n\}$ are independent, then $T$ is 1-to-1. If they span $W$, then $T$ is onto. If they're a basis, $T$ is an isomorphism.

Proof: Any element $v = \sum_{k=1}^n a_k \hat{v}_k \in V$ would have to be sent to $(T(x)) = \sum_{k=1}^n a_k T \hat{v}_k$ by linearity; and this also gives a definition of $T$.

If $\{\hat{w}_1, \ldots, \hat{w}_n\}$ is L.I., then $\forall \hat{v} \in \ker(T) \Rightarrow 0 = T \hat{v} = \sum_{k=1}^n a_k T \hat{v}_k \Rightarrow a_k = 0 \quad (k = 1, \ldots, n) \Rightarrow \hat{v} = 0$. So $T$ is injective.

If any vector $\hat{w} \in W$ is in the linear span of $\hat{w}_1 = T \hat{v}_1$, $\ldots$, $\hat{w}_n = T \hat{v}_n$, then $\hat{w} = \sum_{k=1}^n a_k T \hat{v}_k = T(\sum_{k=1}^n a_k \hat{v}_k) \in \text{im}(T)$.

Example: Let $T : V \rightarrow P_2$ be the L.T. determined by $T(\hat{i}) = 1$, $T(\hat{j}) = x$, $T(\hat{k}) = x^2$. By the Prop., this is an isomorphism.

Proposition 3: Assume $V, W$ are finite-dimensional. Then

$$\dim V = \dim W \iff V \cong W.$$  

Proof: $(\Leftarrow)$ Suppose $T : V \rightarrow W$ (or vice-versa) is an isomorphism.

Then $\ker(T) = \{0\} \Rightarrow \text{nullity}(T) = 0 \Rightarrow \dim(V) = \text{rank}(T) = \text{dim}(\text{im}(T)) = \dim(W)$. 

$\dim(\text{im}(T)) = W \iff \dim(W)$. 


Suppose \( \dim V = n = \dim W \). Let \( \{ \vec{v}_1, \ldots, \vec{v}_n \} \) and \( \{ \vec{w}_1, \ldots, \vec{w}_n \} \) be bases. The L.T. defined (via Prop 2) to send \( \vec{v}_j \rightarrow \vec{w}_j \) is an isomorphism. \( \square \)

We may compose linear transformations (just as we would compose any functions): e.g.,

\[ U \xrightarrow{R} V \xrightarrow{T} W \]

is written \( T \circ R \) (or \( TR \)). This is linear b/c

\[ TR(\alpha \vec{x} + \beta \vec{y}) = T(R(\alpha \vec{x} + \beta \vec{y})) = T(\alpha R \vec{x} + \beta R \vec{y}) = \alpha TR \vec{x} + \beta TR \vec{y} \]

**Definition:** Given \( T : V \rightarrow W \), \( S : W \rightarrow V \) is

(i) a right inverse for \( T \) if \( T \circ S = I_W \)

(ii) a left inverse for \( T \) if \( S \circ T = I_V \)

(iii) an inverse of \( T \) if it is both: write \( S = T^{-1} \).

(\text{In this case, we say \( T \) is invertible.)}

**Theorem:** \( T : V \rightarrow W \) has ...

(i) a right inverse \( \iff \) \( T \) is onto

(ii) a left inverse \( \iff \) \( T \) is 1-to-1

(iii) an inverse \( \iff \) \( T \) is an isomorphism.

**Proof:**

(i) \((\Rightarrow)\): If \( TS = I_W \), then any \( \vec{w} = I_W(\vec{w}) = T(S\vec{w}) \in \text{im}(T) \).

\((\Leftarrow)\): I prove this only in the finite dimensional case.

(Otherwise it requires a dimensional basis & the Axiom of Choice)

Let \( \{ \vec{w}_1, \ldots, \vec{w}_n \} \) = basis of \( W \) and choose \( \vec{v}_1, \ldots, \vec{v}_n \) with

\[ \vec{v}_j = \vec{w}_j \quad \text{(since \( T \) is onto)}. \]

Define \( S \) to and \( \vec{v}_j \rightarrow \vec{w}_j \).
(\Rightarrow) \ : \ \text{Some comment as above} \ \Rightarrow \ \ker(T) = \{0\}.

\text{(in general)}

(\Leftarrow) \ : \ \text{Any} \ \vec{w} \in W \ \text{is in the image of some} \ \vec{v}_m \in \mathcal{V}, \ \text{since} \ \vec{v} = T\vec{w} \ \text{and} \ \text{the map} \ T \ \text{is one-to-one. Since} \ T \ \text{is one-to-one, this} \ \vec{v}_m \ \text{is unique. So define} \ S\vec{w} := \vec{v}_m \ \text{for each} \ \vec{w}. \ \text{Clearly} \ T(S\vec{w}) = T(\vec{v}_m) = \vec{v} \ \text{by definition, and clearly} \ S(T\vec{v}) = \vec{v} \ \text{as well. So} \ S = T^{-1}. \ \Box

Ex: \ \text{Let} \ a_0, a, \ldots, a_n \in \mathbb{R} \ \text{be distinct, and consider the evaluation map} \ T : P_n \rightarrow V_{n+1},\ \text{where} \ f(t) \mapsto (f(a_0), \ldots, f(a_n)).

\text{Is this invertible? an isomorphism?}

Set \ \sum_{i=0}^{n} \ \frac{t-a_i}{a_i-a_k} = \left\{ \begin{array}{ll}
\prod_{k=0}^{i-1} \frac{t-a_k}{a_i-a_k} & \text{if} \ j \neq i \\
1 & \text{if} \ j = i
\end{array} \right., \ \text{(product of these as} \ k \ \text{runs from} \ 0 \ \text{to} \ n, \ \text{skipping} \ i)\)

Then \ \sum_{i=0}^{n} \ \frac{t-a_i}{a_i-a_k} = \left\{ \begin{array}{ll}
0 & \text{if} \ j \neq i \\
1 & \text{if} \ j = i
\end{array} \right., \ \text{(skipping} \ i)\)

Defining \ S(b_0, \ldots, b_n) := \sum_{i=0}^{n} b_i \ \sum_{i=0}^{n} \ \frac{t-a_i}{a_i-a_k} = \sum_{i=0}^{n} b_i \ \text{if} \ f_i(t) \ (i=0) \ \text{where} \ f_i(t) = \prod_{k=0}^{i-1} \frac{t-a_k}{a_i-a_k} \ \text{and} \ T\sum_{i=0}^{n} b_i \ \text{if} \ f_i(t) = \sum_{i=0}^{n} b_i \ \sum_{i=0}^{n} \ \frac{t-a_i}{a_i-a_k} = \left( \sum_{i=0}^{n} b_i \ \text{if} \ (a_i, \ldots, a_i) \right) = (b_0, b_1, \ldots, b_n).
So \( T \circ S = \text{Id}_{V_n} \Rightarrow S \) is a right inverse \( \Rightarrow T \) is onto. By Rank-Nullity,
\[
\dim V_n = \dim(\ker(T)) + \dim(\text{im}(T))
\]
\[
= n + 1 + \dim(\ker(T))
\]
\[
\Rightarrow \ker(T) = \{0\} \Rightarrow T \text{ is 1-to-1} \Rightarrow T \text{ is an isomorphism.}
\]

Notice that what we've really done here is shown that the function \( \sum_{i=0}^n b_i \frac{f(x)}{x-\alpha_i} \) takes prescribed values \( b_0, \ldots, b_n \) at \( \alpha_0, \ldots, \alpha_n \). This is called Lagrange interpolation.

We conclude with the following observation, which came up in the example just done:

**Proposition 4:** If \( V \) \& \( W \) are of the same finite dimension \( n \), then the following are equivalent for \( T : V \to W \):

(a) \( T \) is onto
(b) \( T \) is 1-to-1
(c) \( T \) is an isomorphism

**Proof:** Use Rank-Nullity:
\[
\dim(\text{im}(T)) + \dim(\ker(T)) = n.
\]
If (a) holds, \( \dim(\text{im}(T)) = \dim(W) = n \Rightarrow \ker = \{0\} \Rightarrow (b) \).
If (b) holds, \( \dim(\ker(T)) = 0 \Rightarrow \dim(\text{im}(T)) = n \Rightarrow \text{im}(T) = W \).

Since (a)+(b) is equivalent to (c), done. \( \Box \)