Lecture 6: Integration of step functions

Let $f : [a, b] \to \mathbb{R}_{\geq 0}$ be a function with domain the closed interval $[a, b] := \{ x \in \mathbb{R} | a \leq x \leq b \}$.

**Definition 1**: The **ordinate set** of $f$ is

$$Q_f := \{ (x, y) \in \mathbb{R}^2 | a \leq x \leq b, 0 \leq y \leq f(x) \}.$$

**Question**: For what functions $f$ is this "well-approximable" — that is, $Q_f \in \mathcal{M}$ and $\sigma(Q_f)$ is defined?

We'll see that the class of such functions is quite large in the next lecture; for now, we will consider a small class of functions. I'd also like to point out that it isn't true for just any function:

**Ex/On** $[0, 1]$, neither $f(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \text{ (x is rational)} \\ 0 & \text{otherwise (x is irrational)} \end{cases}$

nor $g(x) := \begin{cases} 1/x, & x > 0 \\ 0, & x = 0 \end{cases}$ is well-approximable.

For $f$, the smallest step-region enclosing $Q_f$ is $T = [0, 1] \times [0, 1]$, while the largest step-region contained in $Q_f$ is $S = [0, 1] \times \{0\}$.

So there is certainly not a unique real number $c$ satisfying $\sigma(S) \leq c \leq \sigma(T)$ for all step-regions $S, T$ with $S \subseteq Q_f \subseteq T$:

- the largest $\sigma(S)$ can be is 0 and the smallest $\sigma(T)$ can be is 1.

For $g$, we have a different issue: we can choose the "lower step region" $S$ to be the $S_n$ in the picture below.
so that \( a(S_n) = \sum_{j=2}^{n} \frac{1}{(\frac{1}{2})^j} = \sum_{j=2}^{n} \frac{1}{j} \).

Though we won't prove this now, this can be made arbitrarily large by taking \( n \) arbitrarily large. So in this case the problem is not uniqueness of \( c \) but existence of \( c \): there is no number at all which is \( \geq a(S) \) for every step region \( S \) with \( S \subseteq \Omega \).

**Definition 2:** A partition of \([a, b]\) is a finite set
\[
P = \{x_0, x_1, ..., x_n\} \subset \mathbb{R}
\]
of real numbers with \( a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b \).

It determines a subdivision \([a, b] = \bigcup_{i=1}^{n} [x_{i-1}, x_i] \).

**Definition 3:** A step function on \([a, b]\) is a function
\( s : [a, b] \rightarrow \mathbb{R} \) which is constant on each open subinterval \((x_{i-1}, x_i)\) of some partition \( P \) of \([a, b]\). (Write \( s_i \) for this constant value. Note that \( s_i \) may take completely unrelated values at the points \( x_i \).)

**Example**
\( S(x) = [x] = \) greatest integer \( \leq x \). Its graph on \([0, 4]\):

![Graph](image)

*E.g. \([3.91] = 3*.

If we change \( s \) so that \( s(3) = 2.718, 2.81 \), it is still a step function (according to Def. 3).
Note that if \( s \) takes nonnegative values on \([a, b]\), and (for each \( i \)) \( s(x_i) \) is the larger of \( s_i^- \) (the value on \((x_{i-1}, x_i)\)) and \( s_i^+ \) (the value on \((x_i, x_{i+1})\)), then the ordinate set \( Q_s \) is a step region, i.e. a union of closed rectangles. But even if we change the values at the points \( \{x_i\} \), that only adds or subtracts line segments from \( Q_s \), which doesn't affect its area.

**Definition 4:** The definite integral of a step function \( s \) on \([a, b]\) is

\[
\int_a^b s(x) \, dx := \sum_{i=1}^n (x_i - x_{i-1}) \cdot s_i^- (= a(Q_s) \text{ if } s \geq 0 \text{ on } [a, b]).
\]

**Ex** /

\[
\int_0^4 [x] \, dx = (1-0) \cdot 0 + (2-1) \cdot 1 + (3-2) \cdot 2 + (4-3) \cdot 3 = 1 + 2 + 3 = 6.
\]

**Ex** /

\[
\int_0^4 [2x] \, dx = ?
\]

In this case, the value jumps each time we add \( \frac{1}{2} \) to \( x \). So we have to use the finer partition \( P = \{0, \frac{1}{2}, \frac{3}{2}, \ldots, 4\} \) to check \( s(x) = [2x] \) as a step function. The integral is then \( \frac{1}{2} \cdot (0+1+2+3+\cdots+7) = 14 \).

**Ex** /

\[
\int_0^n [t] \, dt = 1 - \left(0^2 + 1^2 + 2^2 + \cdots + (n-1)^2\right) = P_2(n-1) = \frac{n(n-1)(2n-1)}{6}.
\]

"Variable of integration" is just a dummy variable — changing its name does nothing.
But does the definition make sense? Given $S$, there are
tons of partitions $P$ on whose subintervals $s$ is constant: just
add more points to $P$—this is called refining the partition.
This had better not affect the value of $\int_a^b s(x) \, dx$!

Fortunately, it doesn't: inserting $y$ between $x_i-1$ and $x_i$
merely subdivides the rectangle $[x_{i-1}, x_i] \times [0, s_i]$ and "changes"
$(x_i-x_{i-1}) \cdot s_i$ to $(x_i-y) \cdot s_i + (y-x_{i-1}) \cdot s_i$ (i.e. changes nothing!).
We say that $\int_a^b s(x) \, dx$ is "well-defined", i.e. depends only on
$a, b, d$ and $s(x)$ itself.

Properties of the definite integral of a step function:

1. $\int_a^b c \cdot s(x) \, dx = c \cdot \int_a^b s(x) \, dx$ ("homogeneous property")
2. $\int_a^b (s(x)+t(x)) \, dx = \int_a^b s(x) \, dx + \int_a^b t(x) \, dx$ ("additive property")
3. If $s(x) \leq t(x)$ for each $x \in [a, b]$, then $\int_a^b s(x) \, dx \leq \int_a^b t(x) \, dx$ ("comparison property")
4. If $a \leq b \leq c$, then $\int_a^b s(x) \, dx + \int_b^c s(x) \, dx = \int_a^c s(x) \, dx$ ("additivity with respect
to the interval")
5. $\int_a^b s(x) \, dx = \int_{a+c}^{b+c} s(x-c) \, dx$ ("translation invariance")
6. $\int_a^b k \cdot s(x) \, dx = k \int_a^b s(x) \, dx$ ("expansion/contraction property")
(We also set $\int_a^b s(x) \, dx := -\int_b^a s(x) \, dx$ if $a > b$.)

Proofs: You can prove all of these just using the definition, but
a geometric proof is more natural for some of them:

- In (5), the LHS (left-hand side) is $a(\xi)$ and the RHS is $a(\xi + c)$, where $\xi$ is translation by $c$ units to the right. By translation-invariance
  of area, these areas are equal.
proving 4) is part of HW #2. You can also do this one geometrically.

5) if \( s(t) \) has underlying partition \( \{x_0, x_1, \ldots, x_n\} \) then \( s(\frac{k}{n}) \) has partition \( \{kx_0, kx_1, \ldots, kx_n\} \) (of \( [ka, kb] \)). The LHS is \( \sum_{i=1}^{n} (kx_i - kx_{i-1}) \cdot s_i = k \sum_{i=1}^{n} (x_i - x_{i-1}) \cdot s_i = \) the RHS.

6) 7) use the same (important) idea: the common refinement of two partitions \( P \) and \( P' \) means combining all the points: \( P \cup P' \). If \( s \) is constant on subintervals of \( P \), and \( t \) is constant on subintervals of \( P' \), then \( s + t \) are both constant on subintervals of \( P \cup P' = \{y_0, y_1, y_2, \ldots, y_N\} \). So therefore is \( s + t \), and (6) simply reduces to \( \sum_{i=1}^{N} (y_i - y_{i-1}) (s_i + t_i) = \sum_{i=1}^{N} (y_i - y_{i-1}) s_i + \sum_{i=1}^{N} (y_i - y_{i-1}) t_i \); while (7) is \( \sum_{i=1}^{N} (y_i - y_{i-1}) s_i < \sum_{i=1}^{N} (y_i - y_{i-1}) t_i \). \( \square \)