Lecture 6: Integration of step functions

Let \( f: [a, b] \to \mathbb{R}_{\geq 0} \) be a function with domain the closed interval \([a, b] := \{ x \in \mathbb{R} | a \leq x \leq b \}\).

**Definition 1:** The ordinate set of \( f \) is \( Q_f := \{ (x, y) \in \mathbb{R}^2 | a \leq x \leq b, 0 \leq y \leq f(x) \} \).

**Question:** For what functions \( f \) is this "well-approximable," hence measurable — that is, \( Q_f \in \mathcal{M} \) and \( a(Q_f) \) is defined?

We'll see that the class of such functions is quite large in the next lecture; for now, we will consider a small class of functions. I'd also like to point out that it isn't true for just any function:

Ex/On \([0, 1]\), neither \( f(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \text{ (x is rational)} \\ 0 & \text{otherwise (x is irrational)} \end{cases} \)

nor \( g(x) := \begin{cases} \frac{1}{x}, & x > 0 \\ 0, & x = 0 \end{cases} \) is well-approximable.

For \( f \), the smallest step-region enclosing \( Q_f \) is \( T = [0, 1] \times [0, 1] \), while the largest step-region contained in \( Q_f \) is \( S = [0, 1] \times \{0\} \).

So there is certainly not a unique real number \( \alpha \) satisfying \( a(S) \leq \alpha \leq a(T) \) for all step-regions \( S, T \) with \( S \subseteq Q_f \subseteq T \):

the largest \( a(S) \) can be is \( 0 \) and the smallest \( a(T) \) can be is \( 1 \).

For \( g \), we have a different issue: we can choose the "lower step region" \( S \) to be the \( S_n \) in the picture below,
so that \( a(S_n) = \sum_{j=2}^{n} \frac{1}{j} = \sum_{j=2}^{n} \frac{1}{j} \).

Though we won't prove this now, this can be made arbitrarily large by taking \( n \) arbitrarily large. So in this case the problem is not uniqueness of \( c \) but existence of \( c \) : there is no number at all which is \( \geq a(S) \) for every step region \( S \) with \( S \subseteq \Omega_g \).

**Definition 2**: A partition of \([a, b]\) is a finite set
\[
P = \{x_0, x_1, \ldots, x_n\} \subseteq \mathbb{R}
\]
of real numbers with \( a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b \).
It determines a subdivision \([a, b] = \bigcup_{i=1}^{n} [x_{i-1}, x_i] \).

**Definition 3**: A step function on \([a, b]\) is a function
\( s : [a, b] \rightarrow \mathbb{R} \) which is constant on each open subinterval \((x_{i-1}, x_i)\) of some partition \( P \) of \([a, b]\). (Write \( s_i \) for this constant value. Note that \( s \) may take completely unrelated values at the points \( x_i \).)

\[Ex/ \ s(x) = \lfloor x \rfloor := \text{greatest integer} \leq x. \text{ Its graph on } [0, 9]:\]

\[\text{e.g. } [3.91] = 3\]

If we change \( s \) so that \( s(3) = 2, 7, 18, 28, 1 \), it is still a step function (according to Def. 3).
Note that if $s$ takes nonnegative values on $[a,b]$, and (for each $i$, $s(x_i)$ is the larger of $S_i$ (the value on $(a_{i-1}, x_i]$) and $S_{i+1}$ (the value on $(x_i, x_{i+1}]$), then the ordinate set $Q_s$ is a step region, i.e., a union of closed rectangles. But even if we change the values at the points $x_i$, that only adds or subtracts line segments from $Q_s$, which doesn't affect its area.

**Definition 4:** The definite integral of a step function $s$ on $[a,b]$ is

$$\int_a^b s(x) \, dx := \sum_{i=1}^n (x_i - x_{i-1}) \cdot S_i \quad (= a(Q_s) \text{ if } s \geq 0 \text{ on } [a,b]).$$

**Ex.** $\int_0^4 [x] \, dx = (1-0) \cdot 0 + (2-1) \cdot 1 + (3-2) \cdot 2 + (4-3) \cdot 3$

$$= 1 + 2 + 3 = 6.$$  //

**Ex.** $\int_0^4 [2x] \, dx = ?$

In this case, the value jumps each time we add $\frac{1}{2}$ to $x$. So we have to use the finer partition $P = \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, 4\}$ to count $S(x) = [2x]$ as a step function. The integral is then $\frac{1}{2} \cdot (0 + 1 + 2 + 3 + \ldots + 7) = 14.$  //

**Ex.** $\int_0^n [t]^2 \, dt = 1 \cdot \left(0^2 + 1^2 + 2^2 + \ldots + (n-1)^2\right) = \frac{p_2(n-1)}{6}$

*Variable of integration* is just a dummy variable — changing its name does nothing.  //
But does the definition make sense? Given \( S \), there are tons of partitions \( P \) on whose subintervals \( S \) is constant: just add more points to \( P \) — this is called refining the partition. This had better not affect the value of \( \int_a^b S(x) \, dx \)!

Fortunately, it doesn’t: inserting \( y \) between \( x_{i-1} \) and \( x_i \) merely subdivides the rectangle \([x_{i-1}, x_i] \times [0, S_i]\) and “changes” \((x_i - x_{i-1})S_i\) to \((x_i - y)S_i + (y - x_{i-1})S_i\) (i.e. changes nothing!). We say that \( \int_a^b S(x) \, dx \) is “well-defined”, i.e. depends only on \( a, b, P \) and \( S(x) \) itself.

**Properties of the definite integral of a step function:**

1. \( \int_a^b c \cdot S(x) \, dx = c \cdot \int_a^b S(x) \, dx \) (“homogeneous property”)
2. \( \int_a^b (S(x) + T(x)) \, dx = \int_a^b S(x) \, dx + \int_a^b T(x) \, dx \) (“additive property”)
3. If \( S(x) < T(x) \) for each \( x \in [a,b] \), then \( \int_a^b S(x) \, dx < \int_a^b T(x) \, dx \) (“comparison property”)
4. If \( a \leq b \leq c \), then \( \int_a^b S(x) \, dx + \int_b^c S(x) \, dx = \int_a^c S(x) \, dx \) (“additivity with respect to the interval”)
5. \( \int_a^{b+c} S(x) \, dx = \int_a^b S(x-c) \, dx \) (“translation invariance”)
6. \( \int_a^b k \cdot S(x) \, dx = k \int_a^b S(x) \, dx \) (“expansion/contraction property”)

(We also set \( \int_b^a S(x) \, dx := -\int_a^b S(x) \, dx \) if \( a < b \).)

**Proofs:** You can prove all of these just using the definition, but a geometric proof is more natural for some of them:

- In (5), the LHS (left-hand side) is \( a(Q_c) \) and the RHS is \( a(c(Q_s)) \),
  - where \( c \) is translation by \( c \) units to the right. By translation invariance of area, these areas are equal.
• proving ④ is part of HW #2. You can also do this one geometrically.

• ⑥: if ⑤ has underlying partition \( \{x_0, x_1, \ldots, x_n\} \) then \( S(k) \) has partition
\[ \{kx_0, kx_1, \ldots, kx_n\} \] (of \([ka, kb]\)). The LHS is
\[ \sum_{i=1}^{n} (kx_i - kx_{i-1}) \cdot s_i = k \sum_{i=1}^{n} (x_i - x_{i-1}) \cdot s_i = \text{the RHS}. \]

• ③ & ⑤ use the same (important) idea: the common refinement of two partitions \( \mathcal{P} \) and \( \mathcal{P}' \) means combining all the points: \( \mathcal{P} \cup \mathcal{P}' \). If \( s \) is constant on subintervals of \( \mathcal{P} \), and \( t \) is constant on subintervals of \( \mathcal{P}' \), then \( s + t \) are both constant on subintervals of \( \mathcal{P} \cup \mathcal{P}' = \{y_0, y_1, y_2, \ldots, y_N\} \). So therefore is \( s + t \), and ⑤ simply reduces to \[ \sum_{i=1}^{N} (y_i - y_{i-1}) (s_i + t_i) = \sum_{i=1}^{N} (y_i - y_{i-1}) s_i + \sum_{i=1}^{N} (y_i - y_{i-1}) t_i; \] while ③ is \[ \sum_{i=1}^{N} (y_i - y_{i-1}) s_i < \sum_{i=1}^{N} (y_i - y_{i-1}) t_i. \]