Lecture 7: Integration of more general functions

We already have some intuition for how this should go. If the function \( f \) is \( \geq 0 \) on \([a,b]\), its integral should be defined to be the area of the region "under the graph of \( f \)". But in some cases, we already know that this region \( Q_f \) has no "area" — either because

\[(1) \ f \ \underline{grows} \ \underline{without} \ \underline{bound} \ \text{somewhere} \ \text{and} \ Q_f \ \underline{is} \ \underline{too} \ "\underline{big}" \ (i.e., \ \text{contains subsets with arbitrarily large area}, \ \text{we find} \ \{x \in \mathbb{R} : x \geq 0\})\]

\[(2) \ f \ \underline{jumps} \ \underline{up} \ \underline{and} \ \underline{down} \ \underline{in \ a \ \underline{crazy} \ \underline{way}} \ \text{and we can't do a good job trapping} \ Q_f \ \underline{between} \ \underline{upper} \ \underline{and} \ \underline{lower} \ \underline{step} \ \underline{regions} \ (\text{like the} \ g(x) = \{1 \ x \in \mathbb{Q} \ \text{example})}.\]

So we will eventually have to assume that \( f \) is "bounded" and "\( f \) doesn't oscillate too much" if we want an integral.

But let's try to make a general

**Definition 1**: Write

\[
\mathcal{S} := \{ \int_a^b s(x) \,dx \mid s \text{ step function}, \ s \leq f \text{ on } [a,b] \}
\]

\[
\mathcal{T} := \{ \int_a^b t(x) \,dx \mid t \text{ step function}, \ f \leq t \text{ on } [a,b] \}
\]

We say \( f \) is integrable on \([a,b]\) if there exists a unique \( I \in \mathbb{R} \) with \( s \leq I \leq t \) for all \( s \in \mathcal{S} \) and \( t \in \mathcal{T} \).
this case, we write \( I = \int_a^b f(x) \, dx \) and call it the integral of \( f \) from \( a \) to \( b \).

**Definition 2:** \( f \) is bounded on \([a, b]\) if there is a \( M \in \mathbb{R}^+ \) such that \(-M \leq f(x) \leq M\) for all \( x \in [a, b] \).

The first thing you may notice is that for \( f \) bounded, \( \mathcal{D} \& \mathcal{T} \) are nonempty. (Why?) This is important!

**Theorem 1:** If \( f \) is bounded, there exists an \( I \in \mathbb{R} \) with \( s \leq I \leq t \) for all \( s \in \mathcal{S} \), \( t \in \mathcal{T} \). (But it may not be unique!)

**Proof:** For all \( s, t \) step functions with \( s \leq f \leq t \) on \([a, b]\), we have \( s = \int_a^b s(x) \, dx \leq \int_a^b t(x) \, dx = t \). So any \( s \in \mathcal{S} \) is a lower bound for \( I \Rightarrow \) the "upper integral" \( \overline{I}(f) := \inf_{s \in \mathcal{S}} I(s) \) exists and \( s \leq \overline{I}(f) \). Hence \( \overline{I}(f) \) is an UB for \( \mathcal{S} \), and the "lower integral" \( \underline{I}(f) := \sup_{t \in \mathcal{T}} I(t) \) exists and \( \underline{I}(f) \leq \overline{I}(f) \). Picking any \( I \in \overline{I}(f) \), we see that \( I \leq \overline{I}(f) \leq \overline{I}(f) \) and \( I \leq \overline{I}(f) \leq I \) for all \( s \in \mathcal{S} \), \( t \in \mathcal{T} \).

Boundness solves issue (1). But we still have to deal with (2):

**Definition 3:** On any interval \( I \) (e.g. \([a, b], [a, b), (a, b], (a, b)\)), we say \( f \) is increasing if \( x < y \Rightarrow f(x) < f(y) \). (It's called "strictly increasing" if \( f(x) < f(y) \).) If \( f \) is either increasing or decreasing on \( I \), it is said to be monotonic there. Finally, \( f \) is piecewise monotonic on \([a, b]\) if there is a partition \( P = \{y_0, y_1, \ldots, y_n\} \) and \( f \) is monotone on each open interval \((y_{i-1}, y_i)\).

**Theorem 2:** \( f \) is integrable on \([a, b]\) if it is bounded and piecewise monotonic.
Proof: First, suppose \( f \) is increasing on \([a,b]\). Since \( f \) is bounded, we must show that \( \underline{I}(f) = \overline{I}(f) \) to make \( I \) unique. Take the partition \( P = \{x_0, x_1, \ldots, x_n\} \) with \( x_i = a + \frac{b-a}{n} i \). Consider the step functions \( s_n(x) = f(x_{i-1}) \) for \( x \in [x_{i-1}, x_i) \) on \([a,b]\), which satisfy \( s_n \leq f \leq t_n \): this means

\[
\begin{align*}
\underline{\lambda}_n &:= \int_a^b s_n(x) \, dx = \frac{b-a}{n} \sum_{i=1}^{n} f(x_{i-1}) \in \mathcal{R} \\
\overline{\lambda}_n &:= \int_a^b t_n(x) \, dx = \frac{b-a}{n} \sum_{i=1}^{n} f(x_i) \in \mathcal{R}
\end{align*}
\]

so that

\[
\underline{\lambda}_n \leq \underline{I}(f) \leq \overline{I}(f) \leq \overline{\lambda}_n
\]

\[\Rightarrow 0 \leq \overline{I}(f) - \underline{I}(f) \leq \overline{\lambda}_n - \underline{\lambda}_n = \frac{b-a}{n} \left( \frac{1}{n} \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \right)
\]

\[\Rightarrow \overline{I}(f) - \underline{I}(f) \leq \frac{b-a}{n} \left( f(b) - f(a) \right) = \frac{C}{n}
\]

\[\Rightarrow \overline{I}(f) = \underline{I}(f), \text{ done.}
\]

This part of the proof is less important (and not so well written.)

Next, if we change the values at \( a \) and \( b \), this means that we have to modify \( s_n(x) \) and \( t_n(x) \) on \([x_0, x_i)\) and \([x_{n-1}, x_n]\), to (say) \( s_a, t_a, s_b, t_b \). So \( \overline{\lambda}_n - \underline{\lambda}_n \) becomes

\[\frac{b-a}{n} \left( t_a - s_a + \sum_{i=2}^{n-1} (f(x_i) - f(x_{i-1})) + t_b - s_b \right) =
\]

\[\frac{b-a}{n} \left( t_a - s_a + t_b - s_b + f(x_{n-1}) - f(x_1) \right) \leq \frac{b-a}{n} (k + 2M) = \frac{C}{n},
\]

\[k \text{ const.}
\]

Same story. So we only need \( f \) increasing on \((a,b)\). Similarly,
The case of \( f \) decreasing is dealt with.

Finally, if \( f \) is only piecewise monotonic, it is integrable on each open subinterval \((y_{i-1}, y_i)\) of the partition, by the last 2 paragraphs. That is, the upper \& lower integrals agree: \( \overline{I}_j(f) = \underline{I}_j(f) \).

But \( \overline{I}(f) \) (for example) is the sup of sums of sub integrals over the subintervals, which is the same as the sum of the sups, \( \sum_j \overline{I}_j(f) \).

Similarly, \( \underline{I}(f) = \sum_j \underline{I}_j(f) \). So \( \overline{I}(f) = \overline{I}(f) \).

\[ \square \]

You can see from the diagram in the proof that, if \( f \geq 0 \), we have \( Q_{f_n} \subseteq Q_f \subseteq Q_{f_n} \), and that only one number — namely \( I = \overline{I}(f) = \overline{I}(f) = \int_a^b f(x) \, dx \) — belongs to \([a(Q_{f_n}), \alpha(Q_{f_n})] = [\alpha_n, \alpha_n] \) for all \( n \in \mathbb{N} \). Hence \( Q_f \in \mathcal{M} \) and \( \alpha(Q_f) = I \).

**Properties (to be proved next week):** For piecewise monotonic, bounded \( f, g \):

1. \( \int_a^b (\alpha f(x) + \beta g(x)) \, dx = \alpha \int_a^b f(x) \, dx + \beta \int_a^b g(x) \, dx \)
2. \( \int_a^b f(x) \, dx + \int_a^b f(x) \, dx = \int_a^b f(x) \, dx \)
3. \( \int_a^b f(x) \, dx = \int_{a+c}^{b+c} f(x-c) \, dx \) (Also: \( \int_a^b f(x) \, dx = -\int_b^a f(x) \, dx \))
4. \( \int_a^b f(x) \, dx = \frac{1}{h} \int_{a+kh}^{b+kh} f(x) \, dx \)
5. \( \int_a^b g(x) \, dx \leq \int_a^b f(x) \, dx \) if \( g \leq f \) on \([a,b] \).
6. [Key consequence of Thm 2 & its proof] If \( f \) is \{bounded on \([a,b]\)\}, then \( \int_a^b f(x) \, dx \) is the (unique) number \( I \) satisfying \( \frac{b-a}{n} \sum_{i=0}^{n-1} f(x_i) \leq I \leq \frac{b-a}{n} \sum_{i=1}^{n} f(x_i) \) for all \( n \in \mathbb{N} \), where \( x_i = a + \frac{b-a}{n} i \).
Ex. 1/  Let \( f(x) = x^p \). You showed in HW#1 that

\[
\frac{b}{n} \sum_{i=0}^{n-1} \left( \frac{bi}{n} \right)^p \leq \frac{b^{p+1}}{p+1} \leq \frac{1}{n} \sum_{i=1}^{n} \left( \frac{bi}{n} \right)^p
\]

for every \( n \). So \( \int_a^b x^p \, dx = \frac{b^{p+1}}{p+1} \). More generally,

\[
\int_a^b x^p \, dx = \int_a^0 x^p \, dx + \int_0^b x^p \, dx = \int_0^b x^p \, dx - \int_0^a x^p \, dx
\]

\[
= \frac{b^{p+1} - a^{p+1}}{p+1}.
\]

//

Ex 2/  \[ \int_0^5 x^2 (x-5)^4 \, dx = \int_{-5}^5 (x+5)^2 x^4 \, dx \]

\[
= \int_{-5}^0 (x^6 + 10x^5 + 25x^4) \, dx
\]

\[
= \int_{-5}^0 x^6 \, dx + 10 \int_{-5}^0 x^5 \, dx + 25 \int_{-5}^0 x^4 \, dx
\]

\[
Evl \quad \frac{(-5)^7}{7} + 10 \frac{(-5)^6}{6} + 25 \frac{(-5)^5}{5}
\]

\[
= \frac{5^6}{21}.
\]