Lecture 1: Linear systems

Geometric viewpoint

For simplicity, start with systems of $n$ linear equations in $n$ unknowns, where $n=2$ or $3$.

These will have 3 presentations:
(a) by "row" equations
(b) by "column" equation
(c) by "matrix" equation.

Ex 1/ Consider the system (in form (a))

\[
\begin{align*}
2x_1 - 2x_2 &= -1 \\
-x_1 + x_2 &= -1,
\end{align*}
\]

with accompanying picture:

... from which we see that $(x_1, x_2) = (3, 2)$ is the unique solution of the system.
(b) Now we can rewrite the system as
\[ x_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \]
which asks the question:
"Can we produce \((-1)\) as a linear combination of \((-1, 1)\) and \((-2, 1)\)?"

The answer, as shown in the picture, is YES.

(c) The matrix form of the equation is
\[
\begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}
\]
"\[ A \cdot \bar{x} = \bar{b} \]"

The system in the Example is called consistent because a solution exists. Here is an inconsistent one:

Ex 2/ If we change the first equation in Ex. 1 to
\[ 2x_1 - 2x_2 = -1, \]
then picture (a) becomes
\[
\text{(2 parallel lines)}
\]
While in (b) we have
\[ x_1 \left( \begin{array}{c} 2 \\ -1 \end{array} \right) + x_2 \left( \begin{array}{c} -2 \\ 1 \end{array} \right) = \left( \begin{array}{c} -1 \\ -1 \end{array} \right), \]
which is impossible (as any linear combination of \( \left( \begin{array}{c} 2 \\ -1 \end{array} \right) \) and \( \left( \begin{array}{c} -2 \\ 1 \end{array} \right) \) lies on the line \( L \), and \( (-1) \) does not).

Ex 3/ Finally, if we change \( (-1) \) to something lying on the line \( L \), say \( \left( \begin{array}{c} 4 \\ -2 \end{array} \right) \), then (in (b)) there are many linear combinations of \( \left( \begin{array}{c} 2 \\ -1 \end{array} \right) \) and \( \left( \begin{array}{c} -2 \\ 1 \end{array} \right) \) that will do. Correspondingly, the two parallel lines in picture (a) of Example 2 merge, and we have infinitely many solutions (⇒ consistent).

Turning to \( n = 3 \), here is a consistent example:

Ex 4/ (a) \[ \begin{align*}
3 \text{ planes in space} & \quad \{ x_1 - x_2 - x_3 = 0 \} \\
& \quad \{-x_1 + x_2 - x_3 = 2 \} \\
& \quad \{ x_1 + 2x_3 = 0 \}
\end{align*} \]

Note that two of the planes pass through the origin \((0,0,0)\). (why?)
(b) \[ x_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \]

(c) \[
\begin{pmatrix}
1 & -1 & -1 \\
-1 & 1 & -1 \\
1 & 0 & 2
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} =
\begin{pmatrix}
0 \\
2 \\
0
\end{pmatrix}
\]

Ex 5/ If in Ex. 4(a), we made the last equation \( x_3 = -1 \), we would get a line as solution set.

On the other hand, if we move the \( x_3 = -1 \) plane up or down to \( x_3 = a \) \((\neq -1)\), then we get the picture.
so that there are no common solutions (and the system is inconsistent).

Correspondingly, what happens with the vector equations? The linear combinations

\[ x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \]

include \( \begin{pmatrix} 0 \\ 2 \end{pmatrix} \), but not \( \begin{pmatrix} 2 \\ 2 \end{pmatrix} \) for \( a \neq 1 \).

From these examples, we can (informally) glean that

(i) There are 3 possibilities for linear systems: no solutions, one solution, or infinitely many.

(ii) The \( n \) equations have a common solution (i.e., are consistent) \( \iff \) the column vector on the right-hand-side of the vector equation can be written as a linear combination of the column vectors on the left-hand-side.

(iii) The equations have a common solution for every \( \vec{b} \) \( \iff \) linear combinations of the left-hand-side column vectors fill up all of \( n \)-space.
Algebraic viewpoint

This will be a first glimpse at Gaussian elimination/row operations, to be made more systematic in subsequent lectures.

Ex 6 / Find the solution set (possibly empty) of the system

\[
\begin{align*}
(p_1) & \quad x_1 + x_2 + x_3 = 9 \\
(p_2) & \quad 2x_1 + 4x_2 - 3x_3 = 1 \\
(p_3) & \quad 3x_1 + 6x_2 - 5x_3 = 0
\end{align*}
\]

Eliminate \( x_1 \) in last 2 equations:

\[
\begin{align*}
(p_2) & \rightarrow p_2 - 2p_1 \\
(p_3) & \rightarrow p_3 - 3p_1
\end{align*}
\]

New system:

\[
\begin{align*}
(p_1) & \quad x_1 + x_2 + x_3 = 9 \\
(p_2) & \quad 2x_2 - 5x_3 = -17 \\
(p_3) & \quad 3x_2 - 8x_3 = -27
\end{align*}
\]

Eliminate \( x_2 \) in last equation:

\[
\begin{align*}
(p_3) & \rightarrow p_3 - \frac{3}{2}p_2
\end{align*}
\]

New system:

\[
\begin{align*}
(p_1) & \quad x_1 + x_3 = 9 \\
(p_2) & \quad 2x_2 - 5x_3 = -17 \\
(p_3) & \quad -\frac{1}{2}x_3 = -\frac{3}{2}
\end{align*}
\]

So \( x_3 = 3 \), and back-substituting in (p₁) gives \( 2x_2 - 15 = 17 \) \( \Rightarrow x_2 = -1 \), where you substituting in (p₁) gives \( x_1 + 3 = 9 \) \( \Rightarrow x_1 = 7 \). You can check that \((7, -1, 3)\) solves the original system.
Why does this work? Applying
Elementary Row Operations
(a) Replace an equation/row by itself + multiples of other equations/rows
(b) Swap two equations/rows
(c) Scale an equation/row (multiply by a nonzero constant)

produces a new "row-equivalent" system of equations whose solution set certainly includes all the old solutions. In fact, since (a)-(c) are reversible, the new solution set is the same:

Row-equivalent systems are equivalent.

Here's an example that uses all 3 operations:

Ex 7 /
\[
\begin{align*}
2x_1 + 4x_2 + 2x_3 + 4x_4 &= 2 \\
2x_1 + 4x_2 + 3x_3 + 3x_4 &= 3 \\
3x_1 + 6x_2 + 6x_3 + 3x_4 &= 6
\end{align*}
\]

\[
\begin{bmatrix}
2 & 4 & 2 & 4 & 2 \\
0 & 1 & -1 & -1 & -1 \\
2 & 4 & 3 & 3 & 3 \\
3 & 6 & 6 & 3 & 6
\end{bmatrix}
\]
\[ \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & -3 & 3 \end{bmatrix} \xrightarrow{C_{17}: \ p_3 \rightarrow p_3 - p_2} \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix} \]

STOP. The 3rd line corresponds to the equation $0 = 2$, and so the system is inconsistent (no solutions).