Lecture 10: Determinants (concl.)

Determinants and Volume

Let $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n$. What is the "n-volume" of

$$P = P(\vec{v}_1, \dots, \vec{v}_n) := \left\{ a_1\vec{v}_1 + \dots + a_n\vec{v}_n \;\middle|\; 0 \le a_i \le 1 \text{ (for each } i) \right\}?$$

A basic observation is that if you <u>swap</u> $\vec{v}_i$ & $\vec{v}_j$ it doesn't change the volume; while if you <u>scale</u> (multiply) the length of $\vec{v}_i$ by μ, it multiplies the volume by $|\mu|$. Finally, if you <u>replace</u> $\vec{v}_j$ by $\vec{v}_j + a\vec{v}_i$, this causes a shear of the parallelepiped P, and shears don't affect volume. But these are precisely the effects that these operations have on $|\det(A)|$ (the absolute value of $\det(A)$), where $A = \begin{pmatrix} \leftarrow \vec{v}_1 \rightarrow \\ \vdots \\ \leftarrow \vec{v}_n \rightarrow \end{pmatrix}$.

More precisely, if a sequence of row operations gets you from $\vec{e}_1, \dots, \vec{e}_n$ to $\vec{v}_1, \dots, \vec{v}_n$ (as rows of a matrix), and involves scaling by $\mu_1, \dots, \mu_m$ and N swaps, then $\det A = (-1)^N \cdot \prod_{i=1}^{m} \mu_i$. Now $\mathrm{vol}(P(\vec{e}_1, \dots, \vec{e}_n)) = 1$, so $\mathrm{vol}(P(\vec{v}_1, \dots, \vec{v}_n)) = 1 \cdot \prod_{i=1}^{m} |\mu_i|$ by the above observations, and this is $|\det A|$.

Theorem 1: $\text{Vol}\{P(\vec{v}_1,\dots,\vec{v}_n)\} = \left|\det\begin{pmatrix} \leftarrow & \vec{v}_1 & \rightarrow \\ & \vdots & \\ \leftarrow & \vec{v}_n & \rightarrow \end{pmatrix}\right|$

$$= \left|\det\begin{pmatrix} \uparrow & & \uparrow \\ \vec{v}_1 & \cdots & \vec{v}_n \\ \downarrow & & \downarrow \end{pmatrix}\right|.$$

transpose

Now let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with matrix M.

Corollary: If $\Omega \subset \mathbb{R}^n$ is any bounded measurable subset,

$$\frac{\text{Vol}\{T(\Omega)\}}{\text{Vol}\{\Omega\}} = |\det(M)|.$$

"dilation factor"

Sketch: Covering Ω with little parallelopipeds and taking a limit as their size $\to 0$, we see it suffices to check the result for such parallelopipeds:

$$\frac{\text{Vol}\{T(P(\vec{v}_1,\dots,\vec{v}_n))\}}{\text{Vol}\{P(\vec{v}_1,\dots,\vec{v}_n)\}} = \frac{\text{vol}\{P(T\vec{v}_1,\dots,T\vec{v}_n)\}}{\text{vol}\{P(\vec{v}_1,\dots,\vec{v}_n)\}}$$

$$= \left|\frac{\det\begin{pmatrix} \uparrow & & \uparrow \\ M\vec{v}_1 & \cdots & M\vec{v}_n \\ \downarrow & & \downarrow \end{pmatrix}}{\det\begin{pmatrix} \uparrow & & \uparrow \\ \vec{v}_1 & \cdots & \vec{v}_n \\ \downarrow & & \downarrow \end{pmatrix}}\right| = \left|\frac{\det\left(M\cdot\begin{pmatrix} \uparrow & & \uparrow \\ \vec{v}_1 & \cdots & \vec{v}_n \\ \downarrow & & \downarrow \end{pmatrix}\right)}{\det\begin{pmatrix}\vec{v}_1 & \cdots & \vec{v}_n\end{pmatrix}}\right|$$

$$= \left| \frac{\det M \cdot \cancel{\det\left(\vec{v}_1 \cdots \vec{v}_n\right)}}{\cancel{\det\left(\vec{v}_1 \cdots \vec{v}_n\right)}} \right| = |\det M| . \qquad \square$$

Ex/ $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 1\}$

$M = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ matrix of $T : \mathbb{R}^2 \to \mathbb{R}^2$

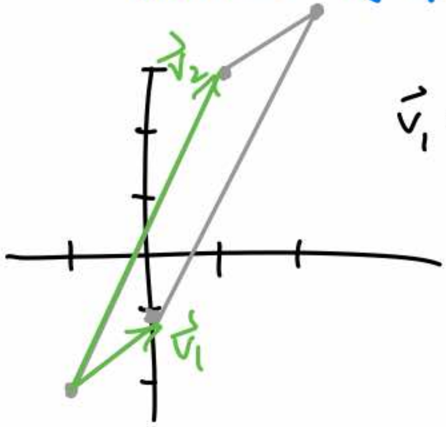
$\Rightarrow T(\Omega) = \left\{(x,y) \in \mathbb{R}^2 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1\right\}$

$\text{area}(T(\Omega)) = \text{area}(\Omega) \cdot |\det M| = \pi ab$. //

[<u>Remark</u>: In calculus, this shows up in the change-of-variable formula for multiple integrals. Given a <u>nonlinear</u> such change, $(x_1, \ldots, x_n) \mapsto (y_1(x_1,\ldots,x_n), \ldots, y_n(x_1,\ldots,x_n))$, the <u>Jacobian</u> matrix is $J = \begin{pmatrix} \partial y_1/\partial x_1 & \cdots & \partial y_1/\partial x_n \\ \vdots & \ddots & \vdots \\ \partial y_n/\partial x_1 & \cdots & \partial y_n/\partial x_n \end{pmatrix}$, and

$|\det J|$ is the "infinitesimal dilation factor" giving the ratio $\frac{dy_1 \wedge \cdots \wedge dy_n}{dx_1 \wedge \cdots \wedge dx_n}$ of volumes of infinitesimal parallelopipeds.]

Ex 2 / Find the area of the parallelogram with vertices $(-1,-2)$, $(1,3)$, $(2,4)$, and $(0,-1)$.



$$\vec{v}_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} - \begin{pmatrix} -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$\text{area}\{P(\vec{v}_1, \vec{v}_2)\} = \left| \det \begin{pmatrix} 1 & 2 \\ 1 & 5 \end{pmatrix} \right| = 3. \quad //$$

$\boxed{\text{Cramer's Rule}}$

Note: Here $A_{\hat{k}}(\vec{b})$ will mean the matrix obtained from A by replacing the k^{th} column by $\vec{b}$.

For $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $A^{-1} = \dfrac{1}{\det A} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$.

So the solution to $A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A^{-1} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \frac{1}{\det A} \begin{pmatrix} a_{22}b_1 - a_{12}b_2 \\ a_{11}b_2 - a_{21}b_1 \end{pmatrix}$$

$$\rightsquigarrow \quad x_1 = \frac{\det \overbrace{\begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix}}^{A_{\hat{1}}(\vec{b})}}{\det A}, \quad x_2 = \frac{\det \overbrace{\begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{pmatrix}}^{A_{\hat{2}}(\vec{b})}}{\det A}.$$

<u>Theorem 2</u>: For A $n \times n$ invertible with columns $\vec{c}_j$, the unique solution of $A\vec{x} = \vec{b}$ is given componentwise by

$$x_i = \frac{\det(A_{\hat{i}}(\vec{b}))}{\det(A)} \quad (i = 1, \dots, n)$$

where $A_i(\vec{b})$ is the $n\times n$ matrix obtained by replacing column $\vec{c}_i$ by $\vec{b}$.

<u>Proof:</u> $\det A_i(\vec{b}) = \det\underbrace{\begin{pmatrix} \uparrow & & \uparrow & & \uparrow \\ \vec{c}_1 & \cdots & \vec{b} & \cdots & \vec{c}_n \\ \downarrow & & \downarrow & & \downarrow \end{pmatrix}}_{i\text{th column}} = \det\begin{pmatrix} \uparrow & & \uparrow & & \uparrow \\ \vec{c}_1 & \cdots & A\vec{x} & \cdots & \vec{c}_n \\ \downarrow & & \downarrow & & \downarrow \end{pmatrix}$ (ith)

$$= \det\underbrace{\begin{pmatrix} \uparrow & & \uparrow & & \uparrow \\ \vec{c}_1 & \cdots & (\sum x_j c_j) & \cdots & \vec{c}_n \\ \downarrow & & \downarrow & & \downarrow \end{pmatrix}}_{i\text{th}} = \sum_{j=1}^{n} x_j \det\underbrace{\begin{pmatrix} \uparrow & & \uparrow & & \uparrow \\ \vec{c}_1 & \cdots & \vec{c}_j & \cdots & \vec{c}_n \\ \downarrow & & \downarrow & & \downarrow \end{pmatrix}}_{\substack{i\text{th} \\ \text{if } j\neq i,\text{ then } \vec{c}_j \\ \text{appears twice} \Rightarrow \det = 0}}$$

$$= x_i \det\begin{pmatrix} \uparrow & & \uparrow & & \uparrow \\ \vec{c}_1 & \cdots & \vec{c}_i & \cdots & \vec{c}_n \\ \downarrow & & \downarrow & & \downarrow \end{pmatrix} = x_i \det A.$$

$\square$

Ex 3/ Solve $\begin{cases} sx - ty = 1 \\ sy - tz = a \\ sz - tx = 0 \end{cases}$ for arbitrary $s,t,a\in\mathbb{R}$ (when possible).

$$\underbrace{\begin{pmatrix} s & -t & 0 \\ 0 & s & -t \\ -t & 0 & s \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 1 \\ a \\ 0 \end{pmatrix}}_{\vec{b}},\qquad \det A = s\begin{vmatrix} s & -t \\ 0 & s \end{vmatrix} - t\begin{vmatrix} -t & 0 \\ s & -t \end{vmatrix}$$

$$= s^3 - t^3.$$

So assume $s^3 \neq t^3$ (or just $s\neq t$, equivalently), so the system is soluble. By Cramer,

$$x = \frac{\begin{vmatrix} 1 & -t & 0 \\ a & s & -t \\ 0 & 0 & s \end{vmatrix}}{\det A} = \frac{s\begin{vmatrix} 1 & -t \\ a & s \end{vmatrix}}{s^3 - t^3} = \frac{s^2 + ast}{s^3 - t^3}$$

$$y = \frac{\begin{vmatrix} s & 1 & 0 \\ 0 & a & -t \\ -t & 0 & s \end{vmatrix}}{s^3 - t^3} = \frac{as^2 + t^2}{s^3 - t^3}, \quad z = \frac{\begin{vmatrix} s & -t & 1 \\ 0 & s & a \\ -t & 0 & 0 \end{vmatrix}}{s^3 - t^3} = \frac{at^2 + st}{s^3 - t^3}.$$

Clearly not something you'd want to do by row-reduction. //

$\boxed{\text{The adjugate matrix}}$

(transpose of the cofactor matrix preferred by Apostol, but more standard)

Let A = invertible $n \times n$. Writing $A^{-1} = \begin{pmatrix} \uparrow & & \uparrow \\ \vec{\alpha}_1 & \cdots & \vec{\alpha}_n \\ \downarrow & & \downarrow \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{pmatrix}$,

$$A \cdot A^{-1} = I_n \implies A \cdot \vec{\alpha}_j = \vec{e}_j \text{ for each } j = 1, \dots, n.$$

By Cramer's rule,

$$\alpha_{ij} = i^{\text{th}} \text{ entry of } \vec{\alpha}_j = \frac{\det\left(A_{\hat{i}}(\vec{e}_j)\right)}{\det(A)}$$

$$A_{\hat{i}}(\vec{e}_j) = \begin{array}{c} \begin{pmatrix} a_{11} & \cdots & 0 & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ \vdots & & 1 & & \vdots \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & 0 & \cdots & a_{nn} \end{pmatrix} \\ (i^{\text{th}} \text{ column}) \end{array}$$

$$\underset{\substack{\text{Laplace} \\ (i^{\text{th}} \text{ column})}}{=} \frac{(-1)^{j+i} \det\left(A_{\hat{j}\hat{i}}\right)}{\det A} = \frac{C_{ji}}{\det(A)}$$

Set $\operatorname{adj}(A) :=$ $n \times n$ matrix with $(i,j)^{\text{th}}$ entry C_{ji}.

<u>Theorem 3</u>: $A^{-1} = \dfrac{1}{\det(A)} \cdot \operatorname{adj}(A)$.

This generalizes the 2×2 formula, though is generally <u>not</u> preferred to the row-reduction approach unless there are variables (or nasty numbers) in the entries of A.

Ex 4/ Find A^{-1} in Example 3.

$$A = \begin{pmatrix} s & -t & 0 \\ 0 & s & -t \\ -t & s & 0 \end{pmatrix} \Rightarrow \quad \det A = s^3 - t^3 \quad \text{and}$$

$$C_{11} = \begin{vmatrix} s & -t \\ 0 & s \end{vmatrix} = s^2, \quad C_{21} = -\begin{vmatrix} -t & 0 \\ 0 & s \end{vmatrix} = st, \quad C_{31} = \begin{vmatrix} -t & 0 \\ s & -t \end{vmatrix} = t^2$$

$$C_{12} = -\begin{vmatrix} 0 & -t \\ -t & s \end{vmatrix} = t^2, \quad C_{22} = \begin{vmatrix} s & 0 \\ -t & s \end{vmatrix} = s^2, \quad C_{32} = -\begin{vmatrix} s & 0 \\ 0 & -t \end{vmatrix} = st$$

$$C_{13} = \begin{vmatrix} 0 & s \\ -t & 0 \end{vmatrix} = st, \quad C_{23} = -\begin{vmatrix} s & -t \\ -t & 0 \end{vmatrix} = -t^2, \quad C_{33} = \begin{vmatrix} s & -t \\ 0 & s \end{vmatrix} = s^2$$

$$\Rightarrow \quad A^{-1} = \frac{1}{s^3 - t^3} \begin{pmatrix} s^2 & st & t^2 \\ t^2 & s^2 & st \\ st & t^2 & s^2 \end{pmatrix}. \quad //$$