An Example (population dynamics)

We begin with a linear system of difference equations (or "discrete dynamical system")

\[ W(t+1) = 0.86 \cdot W(t) + 0.08 \cdot S(t) \]
\[ S(t+1) = -0.12 \cdot W(t) + 1.14 \cdot S(t) \]

describing the wolf & sheep populations over time (measured in months). Set

\[ \mathbf{x}(t) = \begin{pmatrix} W(t) \\ S(t) \end{pmatrix}, \quad A = \begin{pmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{pmatrix}, \]

so that

\[ \mathbf{x}(t+1) = A \cdot \mathbf{x}(t) \]

To understand the system's long-term behavior, we need to compute

\[ \mathbf{x}(t) = A^t \mathbf{x}_0 = A \cdot \ldots \cdot A \left( \begin{pmatrix} W_0 \\ S_0 \end{pmatrix} \right) \]

for very large \( t \).

- Consider the initial state vector \( W_0 = 100 \), \( S_0 = 300 \)
  \[ \mathbf{x}(1) = A \mathbf{x}_0 = \begin{pmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{pmatrix} \begin{pmatrix} 100 \\ 300 \end{pmatrix} = \begin{pmatrix} 110 \\ 330 \end{pmatrix} = 1.1 \cdot \mathbf{x}_0 \]
  Then \( \mathbf{x}(t) = A^t \mathbf{x}_0 = (1.1)^t \mathbf{x}_0 \Rightarrow \) populations grow together.

- Consider the initial state vector \( W_0 = 200 \), \( S_0 = 100 \).
\[ \frac{d\mathbf{x}(t)}{dt} = A \mathbf{x}(t) = \begin{pmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{pmatrix} \begin{pmatrix} 100 \\ 90 \end{pmatrix} = \begin{pmatrix} 180 \\ 90 \end{pmatrix} = 0.9 \mathbf{x}(t) \]

and \[ \frac{d\mathbf{x}(t)}{dt} = A^2 \mathbf{x}(t) = (0.9)^2 \mathbf{x}(t) \Rightarrow \text{populations shrink together: too many wolves eating too few sheep.} \]

Put \[ \mathbf{v}_1 = \begin{pmatrix} 100 \\ 300 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 200 \\ 100 \end{pmatrix} \] → "eigenvalues"

\[ \lambda_1 = 1.1, \quad \lambda_2 = 0.9. \] → "eigenvalues"

Now, in your backyard, \[ \mathbf{x}_0 = \begin{pmatrix} 1000 \\ 1000 \end{pmatrix}. \] So

\[ \frac{d\mathbf{x}(t)}{dt} = A \mathbf{x}_0 = \begin{pmatrix} 940 \\ 1020 \end{pmatrix}, \]

and we aren't in one of the neat cases above. But we can still use those cases to understand what is going on in the long term: break \[ \mathbf{x}_0 = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 \Rightarrow \]

\[ \frac{d\mathbf{x}(t)}{dt} = A \mathbf{x}_0 = c_1 A \mathbf{v}_1 + c_2 A \mathbf{v}_2 = c_1 (1.1) \mathbf{v}_1 + c_2 (0.9) \mathbf{v}_2 \]

and \[ \frac{d\mathbf{x}(t)}{dt} = c_1 (1.1)^t \mathbf{v}_1 + c_2 (0.9)^t \mathbf{v}_2. \]

Grows by 10% every month

Shrinks by 10% every month.

So in case 4:

\[ \mathbf{x}_0 = \begin{pmatrix} 1000 \\ 1000 \end{pmatrix} = 2 \begin{pmatrix} 100 \\ 300 \end{pmatrix} + 4 \begin{pmatrix} 200 \\ 100 \end{pmatrix} = 2\mathbf{v}_1 + 4\mathbf{v}_2, \]

we have

\[ \frac{d\mathbf{x}(t)}{dt} = 2(1.1)^t \begin{pmatrix} 100 \\ 300 \end{pmatrix} + 4(0.9)^t \begin{pmatrix} 200 \\ 100 \end{pmatrix}. \]

The flow lines at right depict the path taken by \( \mathbf{x}(t) \). Clearly we need to get the sheep/wolf population ratio below 1:2...
A little theory

Let $A$ be an $n \times n$ matrix.

**Definition**: A nonzero vector $\mathbf{v} \in \mathbb{R}^n$ is called an **eigenvector** of $A$ if

$$A \mathbf{v} = \lambda \mathbf{v}$$

for some scalar $\lambda$. In this case, $\lambda$ is called an **eigenvalue** of $A$, and $\mathbf{v}$ is called an **eigenvector** (of $A$) with eigenvalue $\lambda$.

Let

$$E_\lambda := \{ \mathbf{v} \in \mathbb{R}^n \mid A \mathbf{v} = \lambda \mathbf{v} \}$$

be the set of all eigenvectors with eigenvalue $\lambda$ (together with the zero vector). This is the eigenspace associated to $\lambda$. Now we make the fundamental

**Observation**: $A \mathbf{v} = \lambda \mathbf{v} \iff (A - \lambda \mathbf{I}_n) \mathbf{v} = \mathbf{0}$,

i.e. $E_\lambda = \text{Nul}(A - \lambda \mathbf{I}_n)$, and is therefore a subspace of $\mathbb{R}^n$. In addition,

$\lambda$ is an eigenvalue of $A \iff \text{Nul}(A - \lambda \mathbf{I}_n) \neq \{\mathbf{0}\}$

$\iff A - \lambda \mathbf{I}_n$ is not invertible

$\iff \text{det}(A - \lambda \mathbf{I}_n) = 0$, proving...
**Theorem 1:** The eigenvalues of $A$ are the solutions to the characteristic equation

$$\det (A - \lambda I_n) = 0.$$ 

Suppose $A$ is upper triangular, say

$$A = \begin{pmatrix} \alpha_1 & * & * \\ 0 & \alpha_2 & * \\ 0 & 0 & \alpha_3 \end{pmatrix} \quad [* = \text{anything}]$$

Then $A - \lambda I_3 = \begin{pmatrix} \alpha_1 - \lambda & * & * \\ 0 & \alpha_2 - \lambda & * \\ 0 & 0 & \alpha_3 - \lambda \end{pmatrix}$

$$\Rightarrow \det (A - \lambda I_3) = (\alpha_1 - \lambda)(\alpha_2 - \lambda)(\alpha_3 - \lambda).$$

**Corollary:** The eigenvalues of an upper triangular matrix are the diagonal entries.

**Another Example**

Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Find all the eigenvalues and bases/dimension for each eigenspace.
(A) First solve the characteristic equation

\[
\det(A - \lambda I_3) = \begin{vmatrix}
1 - \lambda & 1 & 1 \\
1 & 1 - \lambda & 1 \\
1 & 1 & 1 - \lambda
\end{vmatrix} = (1 - \lambda) \begin{vmatrix} 1 & 2 \\
1 & 1 - \lambda
\end{vmatrix} + \begin{vmatrix} 0 & 1 \\
0 & 0
\end{vmatrix}
\]

Now do Laplace

\[
= -\lambda \{(1-\lambda)(\lambda-2) - 2.1\} = -\lambda \{\lambda^2 - 3\lambda + 2 - 2\}
\]

\[
= -\lambda^2 (\lambda - 3)
\]

\[
\Rightarrow \text{eigenvalues are 0 & 3.}
\]

(B) Then compute bases for \(E_0 \& E_3\), viewed as null spaces

- For \(E_3 = \text{Nul}(A - 3I_3) = \text{Nul} \begin{pmatrix} 2 & 1 \\
1 & -2 \\
1 & -2 \\
\end{pmatrix}\)

\[
\text{Row-reduce: } \begin{pmatrix} -2 & 1 & 1 \\
1 & -2 & 1 \\
1 & -2 & 1 \\
\end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 1 \\
0 & 3 & 3 \\
0 & 3 & 3 \\
\end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\
0 & 1 & 1 \\
0 & 0 & 0 \\
\end{pmatrix}
\]

\[
\Rightarrow \text{basis for } E_3 \text{ is } \{(1, 0, 1)\} \Rightarrow \text{dim } E_3 = 1.
\]

- For \(E_0 = \text{Nul}(A - 0I_3) = \text{Nul} \begin{pmatrix} 1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{pmatrix}\)

\[
\text{Row-reduce: } \begin{pmatrix} 1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
\]

\[
\Rightarrow \text{basis for } E_0 \text{ is } \{(1, 0, 1), (0, 1, 1)\} \Rightarrow \text{dim } E_0 = 2.
\]
And in fact, putting the two bases together gives a basis for \( \mathbb{R}^3 \). This doesn't always happen. In the next lecture we'll see why it does happen in some cases (including here).

How does all this relate to a linear transformation \( T : V \to V \)? Eigenvectors \( \vec{v} \) and eigenvalues \( \lambda \in \mathbb{C} \) such that \( T \vec{v} = \lambda \vec{v} \). If \( \{ \vec{b}_1, \ldots, \vec{b}_n \} = \mathcal{B} \) is a basis of \( V \) consisting of eigenvectors with eigenvalues \( \lambda_1, \ldots, \lambda_n \), then

\[
D := [T]_\mathcal{B} = \begin{pmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
& & \ddots & \ddots \\
0 & 0 & \cdots & \lambda_n
\end{pmatrix} = \text{diag} \{ \lambda_1, \ldots, \lambda_n \}
\]

is a diagonal matrix. If \( A \) is some other basis and \( A := [T]_\mathcal{A} \), while \( P := \mathcal{A}^{-1} \mathcal{B} \) is the change-of-basis matrix, then we get

\[(*) \quad P^\top A P = \mathcal{B}^{-1} \mathcal{A} \cdot [T]_\mathcal{A} \cdot \mathcal{A}^{-1} \mathcal{B} = [T]_\mathcal{B} = D.\]

Typically, \( V = \mathbb{R}^n \) and \( \mathcal{A} = \mathcal{E} \) is the standard basis, so that \( T \) is the transformation sending \( \vec{x} \mapsto A \vec{x} \); and \((*)\) says that by writing this \( T \) with respect to an eigenbasis, we see that \( A \) is similar to a diagonal matrix — a process called diagonalizing \( A \). (to be continued...)

(1) Given 2 eigenvectors of \( A \), is their sum an eigenvector?

(2) In each case, is \( \vec{v} \) an eigenvector of \( A \)? If so, find the eigenvalue:

(a) \( \vec{v} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \quad A = \begin{pmatrix} -3 & 1 \\ -3 & 8 \end{pmatrix} \);
(b) \( \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad A = \begin{pmatrix} 4 & -1 \\ -1 & 9 \end{pmatrix} \)

(3) Find the eigenvalues and eigenvectors of \( A = \begin{pmatrix} 7 & 3 \\ 3 & -1 \end{pmatrix} \).