# Lecture 11: Eigenstuff

## An Example (population dynamics)

We begin with a linear system of difference equations (or "discrete dynamical system")

$$W(\tau+1) = 0.86 \cdot W(\tau) + 0.08 \cdot S(\tau)$$

$$S(\tau+1) = -0.12 \cdot W(\tau) + 1.14 \cdot S(\tau)$$

describing the wolf & sheep populations over time (measured in months). Set

$$\vec{x}(\tau) = \begin{pmatrix} W(\tau) \\ S(\tau) \end{pmatrix}, \quad A = \begin{pmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{pmatrix},$$

so that

$$\vec{x}(\tau+1) = A \cdot \vec{x}(\tau).$$

To understand the system's long-term behavior, we need to compute

$$\vec{x}(\tau) = A^{\tau} \vec{x}_0 = \underbrace{A \cdot \cdots \cdot A}_{\tau \text{ times}} \begin{pmatrix} W_0 \\ S_0 \end{pmatrix}$$

← starting values at time $\tau = 0$

for very large $\tau$.

- Consider the initial state vector $W_0 = 100$, $S_0 = 300$

$$\vec{x}(1) = A\vec{x}_0 = \begin{pmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{pmatrix} \begin{pmatrix} 100 \\ 300 \end{pmatrix} = \begin{pmatrix} 110 \\ 330 \end{pmatrix} = 1.1 \cdot \vec{x}_0$$

  Then $\vec{x}(\tau) = A^{\tau}\vec{x}_0 = (1.1)^{\tau}\vec{x}_0 \implies$ populations grow together.

- Consider the initial state vector $W_0 = 200$, $S_0 = 100$.

$$\vec{x}(1) = A\vec{x}_0 = \begin{pmatrix}0.86 & 0.08\\ -0.12 & 1.14\end{pmatrix}\begin{pmatrix}200\\100\end{pmatrix} = \begin{pmatrix}180\\90\end{pmatrix} = 0.9\,\vec{x}_0$$

and $\vec{x}(\tau) = A^{\tau}\vec{x}_0 = (0.9)^{\tau}\vec{x}_0 \implies$ populations shrink together: too many wolves eating too few sheep.

Put $\vec{v}_1 = \begin{pmatrix}100\\300\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}200\\100\end{pmatrix}$ → "eigenvectors"

$\lambda_1 = 1.1$, $\lambda_2 = 0.9$. → "eigenvalues"

Now, in your backyard, $\vec{x}_0 = \begin{pmatrix}1000\\1000\end{pmatrix}$. So

$$\vec{x}(1) = A\vec{x}_0 = \begin{pmatrix}940\\1020\end{pmatrix},$$

and we aren't in one of the neat cases above. But we can still use those cases to understand what is going on in the long term: break $\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2 \implies$

$$\vec{x}(1) = A\vec{x}_0 = c_1 A\vec{v}_1 + c_2 A\vec{v}_2 = c_1(1.1)\vec{v}_1 + c_2(0.9)\vec{v}_2$$

and

$$\vec{x}(\tau) = \underbrace{c_1(1.1)^{\tau}\vec{v}_1}_{\text{grows by 10\% every month}} + \underbrace{c_2(0.9)^{\tau}\vec{v}_2}_{\text{shrinks by 10\% every month}}.$$

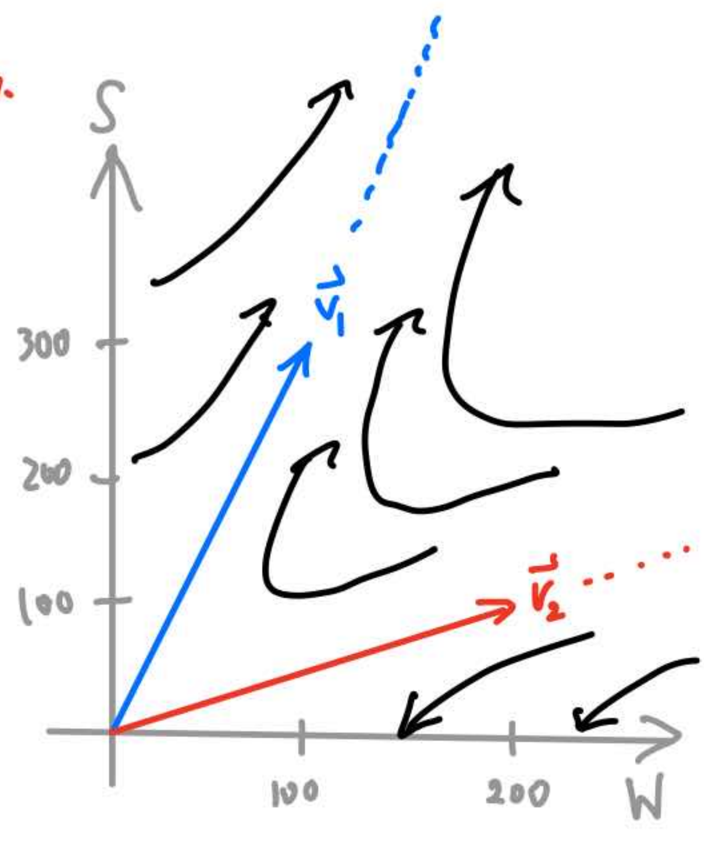
So in case 4

$$\vec{x}_0 = \begin{pmatrix}1000\\1000\end{pmatrix} = 2\begin{pmatrix}100\\300\end{pmatrix} + 4\begin{pmatrix}200\\100\end{pmatrix} = 2\vec{v}_1 + 4\vec{v}_2,$$

we have

$$\vec{x}(\tau) = 2(1.1)^{\tau}\begin{pmatrix}100\\300\end{pmatrix} + 4(0.9)^{\tau}\begin{pmatrix}200\\100\end{pmatrix}.$$

The flow lines at right depict the path taken by $\vec{x}(\tau)$. Clearly we need to get the sheep/wolf population ratio below 1:2 . . .



## A little theory

"eigen" means "inherent" — and indeed, these vectors & values are inherent in the matrix

Let $A$ be an $n \times n$ matrix.

**Definition**: A nonzero vector $\vec{v} \in \mathbb{R}^n$ is called an eigenvector of $A$ if

$$A\vec{v} = \lambda \vec{v}$$

for some scalar $\lambda$. In this case, $\lambda$ is called an eigenvalue of $A$, and $\vec{v}$ is called an eigenvector (of $A$) with eigenvalue $\lambda$.

Let

$$E_\lambda := \{ \vec{v} \in \mathbb{R}^n \mid A\vec{v} = \lambda \vec{v} \}$$

be the set of all eigenvectors with eigenvalue $\lambda$ (together with the zero vector). This is the eigenspace associated to $\lambda$. Now we make the fundamental

**Observation**: $A\vec{v} = \lambda\vec{v} \iff (A - \lambda I_n)\vec{v} = \vec{0}$,

i.e. $$E_\lambda = \mathrm{Nul}(A - \lambda I_n),$$ and is therefore a subspace of $\mathbb{R}^n$! In addition,

$$\begin{aligned} \lambda \text{ is an eigenvalue of } A &\iff \mathrm{Nul}(A-\lambda I_n) \neq \{\vec{0}\} \\ &\iff A - \lambda I_n \text{ is } \underline{\text{not}} \text{ invertible} \\ &\iff \det(A - \lambda I_n) = 0, \end{aligned}$$

proving

**Theorem 1:** The eigenvalues of $A$ are the solutions to the characteristic equation

$$\det(A - \lambda I_n) = 0.$$

Suppose $A$ is upper triangular, say

$$A = \begin{pmatrix} \alpha_1 & * & * \\ 0 & \alpha_2 & * \\ 0 & 0 & \alpha_3 \end{pmatrix}. \qquad [* = \text{anything}]$$

Then $A - \lambda I_3 = \begin{pmatrix} \alpha_1 - \lambda & * & * \\ 0 & \alpha_2 - \lambda & * \\ 0 & 0 & \alpha_3 - \lambda \end{pmatrix}$

$$\Rightarrow \det(A - \lambda I_3) = (\alpha_1 - \lambda)(\alpha_2 - \lambda)(\alpha_3 - \lambda).$$

**Corollary:** The eigenvalues of an upper triangular matrix are the diagonal entries.

## Another Example

Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Find all the eigenvalues and bases/dimension for each eigenspace.

## (A) First solve the characteristic equation

$$\det(A-\lambda I_3) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} \quad \text{subtract} \quad = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 0 & \lambda & -\lambda \end{vmatrix} \quad \text{add}$$

$$= \begin{vmatrix} 1-\lambda & 1 & 2 \\ 1 & 1-\lambda & 2-\lambda \\ 0 & \lambda & 0 \end{vmatrix} = -\lambda \begin{vmatrix} 1-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix}$$

now do Laplace

$$= -\lambda\{(\lambda-1)(\lambda-2) - 2\cdot 1\} = -\lambda\{\lambda^2 - 3\lambda \cancel{+2-2}\}$$

$$= -\lambda^2(\lambda-3)$$

$\Rightarrow$ eigenvalues are $0$ & $3$.

## (B) Then compute bases for $E_0$ & $E_3$, viewed as null spaces

- For $E_3 = \mathrm{Nul}(A-3I_3) = \mathrm{Nul}\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$

row-reduce:
$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_3$ free

$\Rightarrow$ basis for $E_3$ is $\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\} \Rightarrow \dim E_3 = 1$.

- For $E_0 = \mathrm{Nul}(A - 0I_3) = \mathrm{Nul}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

row-reduce:
$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_2, x_3$ free

$\Rightarrow$ basis for $E_0$ is $\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$

$\Rightarrow \dim E_0 = 2$.

And in fact, putting the two bases together gives a basis for $\mathbb{R}^3$. This doesn't always happen. In the next lecture we'll see why it does happen in some cases (including here).

How does all this relate to a linear transformation $T : V \to V$ ? Eigenvectors & eigenvalues are $\vec{v} \in V$ (nonzero!) & $\lambda \in \mathbb{R}$ such that $T\vec{v} = \lambda \vec{v}$. If $\{\vec{b}_1, \ldots, \vec{b}_n\} = \mathcal{B}$ is a basis of $V$ consisting of eigenvectors with eigenvalues $\lambda_1, \ldots, \lambda_n$, then

$$D := [T]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} =: \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}$$

is a diagonal matrix. If $\mathcal{A}$ is some other basis and $A := [T]_{\mathcal{A}}$, while $P := {}_{\mathcal{A}}P_{\mathcal{B}}$ is the change-of-basis matrix, then we get

$$(*) \quad P^{-1} A P = {}_{\mathcal{B}\leftarrow\mathcal{A}}P \cdot [T]_{\mathcal{A}} \cdot {}_{\mathcal{A}\leftarrow\mathcal{B}}P = [T]_{\mathcal{B}} = D.$$

Typically, $V = \mathbb{R}^n$ and $\mathcal{A} = \mathcal{E}$ is the standard basis, so that $T$ is the transformation sending $\vec{x} \mapsto A\vec{x}$; and $(*)$ says that by writing this $T$ with respect to an eigenbasis, we see that $A$ is <u>similar to a diagonal matrix</u> — a process called <u>diagonalizing $A$</u>. (to be continued...)

## Some problems

(1) Given 2 eigenvectors of $A$, is their sum an eigenvector?

(2) In each case, is $\vec{v}$ an eigenvector of $A$? If so, find the eigenvalue:

(a) $\vec{v} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, $A = \begin{pmatrix} -3 & 1 \\ -3 & 8 \end{pmatrix}$ ; (b) $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $A = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$

(3) Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 7 & 3 \\ 3 & -1 \end{pmatrix}$.