

Lecture 15: Quadratic Forms

Definition: A quadratic form on \mathbb{R}^n is a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ of the form $f(\vec{x}) = \vec{x}^T A \vec{x}$ for some symmetric (real, $n \times n$) matrix A called the matrix of the form f .

Ex 1 / Compute $\vec{x}^T A \vec{x}$, for $A = \begin{pmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

$$(x_1, x_2, x_3) \begin{pmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (x_1, x_2, x_3) \begin{pmatrix} 4x_1 + 3x_2 \\ 3x_1 + 2x_2 + x_3 \\ x_2 + x_3 \end{pmatrix} = 4x_1^2 + 2x_2^2 + x_3^2 + 3x_1x_2 + 3x_2x_1 + 6x_1x_2 + x_2x_3 + x_3x_2 + 2x_2x_3$$

In general,

$$(x_1, \dots, x_n) \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (x_1, \dots, x_n) \begin{pmatrix} \sum_{j=1}^n a_{1j} x_j \\ \vdots \\ \sum_{j=1}^n a_{nj} x_j \end{pmatrix} = \sum_{i,j=1}^n x_i a_{ij} x_j$$

$$= \sum_{i=1}^n a_{ii} x_i^2 + \sum_{i < j} a_{ij} x_i x_j + \sum_{i > j} a_{ij} x_i x_j$$

$$= \sum_{i=1}^n a_{ii} x_i^2 + \sum_{i < j} (2a_{ij}) x_i x_j$$

$= \sum_{\substack{\text{sup} \\ \text{idj}}} a_{ji} x_j x_i = \sum_{i < j} a_{ij} x_i x_j$
($A = A^T$)

Ex 2 / Find the matrix of each form:

(on \mathbb{R}^2) • $x_1^2 + x_2^2 = (x_1, x_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{x}^T \mathbb{I}_2 \vec{x} \Rightarrow A = \mathbb{I}_2$

• $x_1 x_2 = (x_1, x_2) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, but $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ isn't symmetric.
 $= \frac{1}{2} x_1 x_2 + \frac{1}{2} x_2 x_1 = (x_1, x_2) \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow A = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$

• $10x_1^2 - 6x_1x_2 - 3x_2^2 = (x_1, x_2) \begin{pmatrix} 10 & -3 \\ -3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow A = \begin{pmatrix} 10 & -3 \\ -3 & -3 \end{pmatrix}$

(on \mathbb{R}^3) • $8x_1^2 + 7x_2^2 - 3x_3^2 - 6x_1x_2 + 4x_1x_3 - 2x_2x_3$

think: $-3x_1x_2 - 3x_2x_1 + 2x_1x_3 + 2x_3x_1 - x_2x_3 - x_3x_2$

$\Rightarrow A = \begin{pmatrix} 8 & -3 & 2 \\ -3 & 7 & -1 \\ 2 & -1 & -3 \end{pmatrix}$

Level sets of quadratic forms

What does the solution set of

$$8x_1^2 - 4x_1x_2 + 5x_2^2 = 1$$

look like? It's an ellipse, but far from obvious you'll need to perform a rotation of coordinates. Begin by recognizing the left-hand side as a quadratic form

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{x}^T A \vec{x} = Q(\vec{x}),$$

where A is symmetric.

But let's step back for a moment and look more generally at the equation

$$(*) \quad \vec{x}^T A \vec{x} = 1 \quad \leftarrow \text{or any positive number}$$

with A symmetric $n \times n$. By the Spectral Theorem (Lecture 14), there is an orthonormal eigenbasis $B = \{\vec{u}_1, \dots, \vec{u}_n\}$ for A , so that $P_B^{-1} = P_B^T$ (why?) and

$$A = P_B \underbrace{\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}}_D P_B^T$$

Writing $\vec{y} = [\vec{x}]_B = P_B^{-1} \vec{x} = P_B^T \vec{x}$ for the "eigencoordinates",

we have

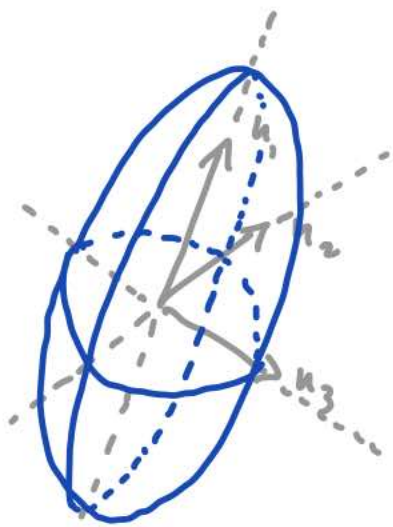
$$\vec{x}^T A \vec{x} = \vec{x}^T P_B D P_B^T \vec{x} = (P_B^T \vec{x})^T D (P_B^T \vec{x}) = \vec{y}^T D \vec{y},$$

and our equation (*) becomes

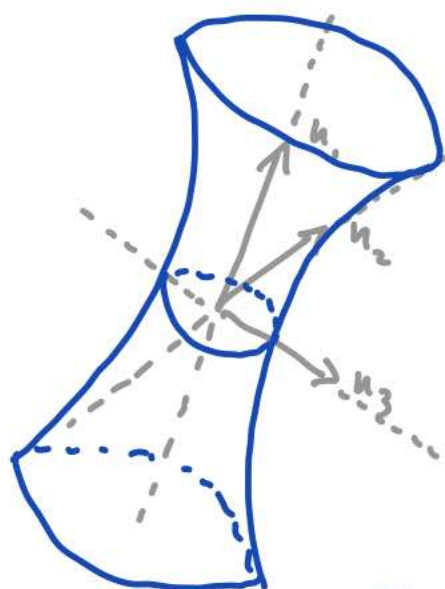
$$(**) \quad \lambda_1 y_1^2 + \dots + \lambda_n y_n^2 = 1.$$

To interpret this geometrically for $n=3$, assume that

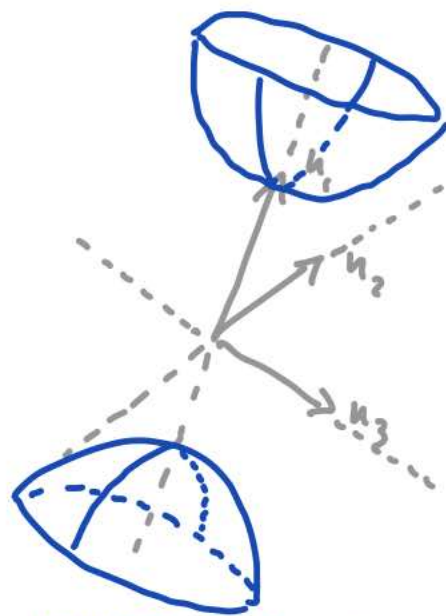
$\lambda_1 \leq \lambda_2 \leq \lambda_3$. If all 3 eigenvalues are positive, then (***) will define an ellipsoid with principal axes (in the directions of $\vec{u}_1, \vec{u}_2, \vec{u}_3$) of lengths $\frac{1}{\sqrt{\lambda_1}}, \frac{1}{\sqrt{\lambda_2}}, \frac{1}{\sqrt{\lambda_3}}$. If only $\lambda_2, \lambda_3 > 0$, then (***) is an elliptic hyperboloid; and if only $\lambda_3 > 0$, (***) is a hyperboloid of 2 sheets.



ELLIPSOID



ELLIPTIC HYPERBOLOID



HYPERBOLOID OF TWO SHEETS

Back to $n=2$, and our original equation: A orthogonally diagonalizes

$$\begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$$

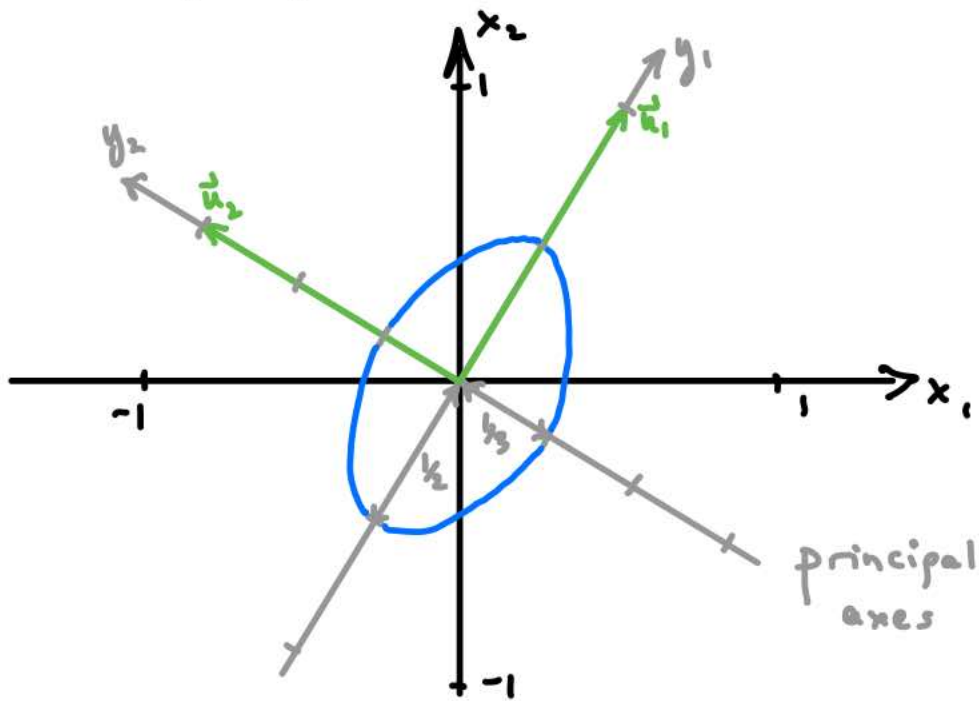
and so if (y_1, y_2) are coordinates along the axes defined by the (unit) eigenvectors

$$\vec{u}_1 = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$$

Our equation becomes

$$1 = 4y_1^2 + 9y_2^2 = \frac{y_1^2}{(1/2)^2} + \frac{y_2^2}{(1/3)^2}.$$

Now it is really easy to sketch the solution:



Q: What sorts of figures do you get if
 $(n=2) \quad \lambda_2 > 0 \quad \text{and} \quad \lambda_1 = 0$
 $(n=3) \quad \lambda_3 > 0, \quad \lambda_2 = 0, \quad \lambda_1 < 0 \quad ?$

Quadratic forms as functions

Suppose $A = PDP^T$ is an orthogonal diagonalization. Then

$$(†) \quad Q(\vec{x}) = \vec{x}^T A \vec{x} = \vec{x}^T P D P^T \vec{x} = (P^T \vec{x})^T D (P^T \vec{x}) \\ = \vec{y}^T D \vec{y} = \lambda_1 y_1^2 + \dots + \lambda_n y_n^2$$

as before. So it's pretty clear that Q is "controlled by the eigenvalues λ_i " once you do the orthogonal change of coordinates.

Definition: (i) Q is positive definite $\iff Q(\vec{x}) > 0 \quad \forall \vec{x} \neq \vec{0}$
positive semidefinite $\iff Q(\vec{x}) \geq 0 \quad \forall \vec{x}$
indefinite $\iff Q$ assumes both positive & negative values on nonzero \vec{x} 's
negative semidefinite $\iff Q(\vec{x}) \leq 0 \quad \forall \vec{x}$
negative definite $\iff Q(\vec{x}) < 0 \quad \forall \vec{x} \neq \vec{0}$

(ii) A is positive definite \iff all $\lambda_i > 0$
positive semidefinite \iff all $\lambda_i \geq 0$
indefinite \iff A has both positive & negative eigenvalues
negative semidefinite \iff all $\lambda_i \leq 0$
negative definite \iff all $\lambda_i < 0$.

Theorem: Q is \iff A is

(Insert same term in both places.)

Proof: Clear from (†). e.g., if λ_i 's are > 0 , then

$$Q(\vec{x}) = \lambda_1 y_1^2 + \dots + \lambda_n y_n^2 \geq 0, \text{ and } = 0 \iff \vec{y} = \vec{0} \iff \vec{x} = \vec{0}. \quad \square$$

Quadratic forms and inner products

Let A be an $n \times n$ symmetric matrix.

Corollary (to the Theorem): $\langle \vec{x}, \vec{y} \rangle := \vec{x}^T A \vec{y}$ is an inner product on $\mathbb{R}^n \iff A$ is positive definite.

Proof: A is positive-definite $\iff Q(\vec{x}) := \langle \vec{x}, \vec{x} \rangle$ is positive-definite
(eigenvalues > 0) Thm. ($\langle \vec{x}, \vec{x} \rangle \geq 0$ with equality $\iff \vec{x} = \vec{0}$)

\iff the symm. bilinear form $\langle \cdot, \cdot \rangle$ satisfies the positive-definiteness property

$\iff \langle \cdot, \cdot \rangle$ is an inner product. □

More geometrically,

$\vec{x}^T A \vec{y}$ gives an inner product

\iff

the set $\|\vec{x}\| = 1$ (comprising elements of "length" 1) is an ellipsoid.