Lecture 16: Operators on inner-product spaces

Recall that an inner product $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$ on a real vector space must be:

$$\text{symmetric} \left\{ \langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle \right.$$

$$\text{linear (in both entries)} \left\{ \begin{array}{l} \langle \vec{x}+\vec{y}, \vec{z} \rangle = \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle \\ \langle c\vec{x}, \vec{z} \rangle = c \langle \vec{x}, \vec{z} \rangle \end{array} \right.$$

$$\text{positive definite} \left\{ \langle \vec{x}, \vec{x} \rangle > 0 \text{ for } \vec{x} \neq \vec{0} \right.$$

If V is a complex vector space, one replaces symmetry by

$$\text{Hermitian symmetry} \left\{ \langle \vec{x}, \vec{y} \rangle = \overline{\langle \vec{y}, \vec{x} \rangle} \right.$$

which implies conjugate linearity in the second entry: $\langle \vec{x}, c\vec{z} \rangle = \bar{c} \langle \vec{x}, \vec{z} \rangle$.
Since $r = \bar{r}$ for real numbers, we can discuss both real & complex at once.

A linear endomorphism (transformation) $T : V \to V$ on an inner product space is often called an operator.

Definition 1: T is self-adjoint $\iff \langle T\vec{x}, \vec{y} \rangle = \langle \vec{x}, T\vec{y} \rangle \; (\forall \vec{x}, \vec{y} \in V)$

skew-adjoint $\iff \langle T\vec{x}, \vec{y} \rangle = -\langle \vec{x}, T\vec{y} \rangle$ "

unitary $\iff \langle T\vec{x}, T\vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle$ "

Remark

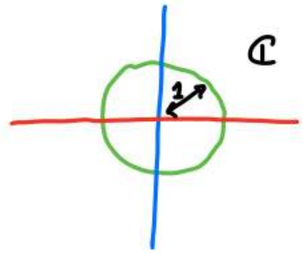
The terminology comes from the notion of the adjoint $T^\dagger$ of T, which is supposed to satisfy $\langle T\vec{x}, \vec{y} \rangle = \langle \vec{x}, T^\dagger \vec{y} \rangle \; (\forall \vec{x}, \vec{y})$. (We will later see that this exists & is unique in the finite-dimensional case.) The three parts of the Defn. correspond to $T = T^\dagger$, $T = -T^\dagger$, and $T^\dagger T = I$. (Here $I : V \to V$ denotes the identity transformation.)

$\square$

Prop. 1: Eigenvalues of a $\begin{cases}\text{self-adjoint}\\ \text{skew-adjoint}\\ \text{unitary}\end{cases}$ operator are $\begin{cases}\text{real}\\ \text{pure imaginary}\\ \text{unit length}\end{cases}$

(means a real # times i)

Proof: If $T\vec{v} = \lambda\vec{v}$, then in the first 2 cases



$$\lambda\langle v,v\rangle = \langle \lambda v, v\rangle = \langle Tv, v\rangle = \pm\langle v, Tv\rangle = \pm\langle v, \lambda v\rangle$$
$$= \pm\bar{\lambda}\langle v,v\rangle \quad \text{and } \langle v,v\rangle \neq 0$$

$\Rightarrow \lambda = \pm\bar{\lambda}$. In the unitary case,

$$\langle v,v\rangle = \langle Tv, Tv\rangle = \langle \lambda v, \lambda v\rangle = \lambda\bar{\lambda}\langle v,v\rangle = |\lambda|^2\langle v,v\rangle \Rightarrow |\lambda|^2 = 1$$
$$\Rightarrow |\lambda| = 1. \qquad \square$$

Prop. 2: Eigenvectors with distinct eigenvalues are orthogonal, in each of the cases in Prop. 1.

Proof: If $T\vec{v} = \lambda\vec{v}$ & $T\vec{w} = \eta\vec{w}$, then in the first 2 cases

$$\lambda\langle\vec{v},\vec{w}\rangle = \langle T\vec{v},\vec{w}\rangle = \pm\langle\vec{v}, T\vec{w}\rangle = \pm\bar{\eta}\langle\vec{v},\vec{w}\rangle = \eta\langle\vec{v},\vec{w}\rangle \text{ and}$$

use Prop. 1!

$\lambda \neq \eta \Rightarrow \langle\vec{v},\vec{w}\rangle = 0$.

In the unitary case, $\langle\vec{v},\vec{w}\rangle = \langle T\vec{v}, T\vec{w}\rangle = \lambda\bar{\eta}\langle\vec{v},\vec{w}\rangle$ and

$$\lambda\bar{\eta} \neq \eta\bar{\eta} = 1 \Rightarrow \langle\vec{v},\vec{w}\rangle = 0. \qquad \square$$

Prop. 1

Ex/ Let V be the infinitely differentiable (possibly complex-valued) functions on $\mathbb{R}$ which are periodic with period 1: $f \in V \Rightarrow f(a+1) = f(a)\ \forall a \in \mathbb{R}$. Define an inner product by $\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}\,dx$, and note that V is closed under differentiation — $T = \frac{d}{dx} : V \to V$.

Then $\langle Tf, g\rangle = \int_0^1 f'(x)\overline{g(x)}\,dx = \underbrace{f\bar{g}\Big|_0^1}_{0} - \int_0^1 f(x)\overline{g'(x)}\,dx$

$= -\langle f, Tg\rangle \Rightarrow T$ is skew-adjoint $\Rightarrow T^2$ is self-adjoint $\left(\langle T^2 f, g\rangle = -\langle Tf, Tg\rangle = \langle f, T^2 g\rangle\right)$.

Since $\vec{v}_k := \sin\left(\frac{kx}{2\pi}\right)$ are eigenvectors with distinct eigenvalues $\lambda_k = -\frac{k^2}{4\pi^2}$ under T^2, they are automatically orthogonal:

$$\int_0^1 \sin\left(\frac{kx}{2\pi}\right)\sin\left(\frac{lx}{2\pi}\right)dx = 0 \quad \text{if } l \neq k.$$

//

<u>Spectral Theorem</u>: Assume that V is finite-dimensional. Then T has an orthonormal eigenbasis.

<u>Proof</u>: Let λ_1 be any root of $0 = \underbrace{\det(\lambda I - T)}$.

write $\lambda I - T$ w.r.t. any basis, take det — independent of choice of basis (why?)

Then an eigenvector $\vec{v}_1$ exists: $T\vec{v}_1 = \lambda_1 \vec{v}_1$. Normalize so $\|\vec{v}_1\| = 1$.

Let $W = \{\vec{w} \in V \mid \langle \vec{w}, \vec{v}_1 \rangle = 0\}$. Then T restricts to W:

$\langle T\vec{w}, \vec{v}_1 \rangle = \pm\langle \vec{w}, T\vec{v}_1 \rangle = \pm\overline{\lambda}_1 \langle \vec{w}, \vec{v}_1 \rangle = 0 \Rightarrow T\vec{w} \in W$.

(In the unitary case, $\vec{v}_1 = \overline{\lambda}_1 \lambda_1 \vec{v}_1 = \overline{\lambda}_1 T\vec{v}_1 \Rightarrow \langle T\vec{w}, \vec{v}_1 \rangle = \langle T\vec{w}, \overline{\lambda}_1 T\vec{v}_1 \rangle$
$= \lambda_1 \langle T\vec{w}, T\vec{v}_1 \rangle = \lambda_1 \langle \vec{w}, \vec{v}_1 \rangle = 0 \Rightarrow T\vec{w} \in W$.) Since $\dim W = \dim V - 1$, proof is done by induction on dimension. □

Ex/ Let $p, q \in C^1[a,b]$, with $p(a) = p(b) = 0$, and $V \subseteq C^\infty[a,b]$ any subspace closed under the operator $Tf := (pf')' + qf$; the inner product is $\langle f, g \rangle := \int_a^b f(x)g(x)\,dx$. We calculate

$$\langle Tf, g \rangle - \langle f, Tg \rangle = \int_a^b \left\{(pf')'g + \cancel{qfg} - \left((pg')'f + \cancel{qgf}\right)\right\}dx$$

$$= \underbrace{pf'g\Big|_a^b}_{0 \text{ b/c } p(a)=0=p(b)} - \cancel{\int_a^b pf'g'\,dx} - \left(\underbrace{pg'f\Big|_a^b}_{0} - \cancel{\int_a^b pg'f'\,dx}\right)$$

$= 0 \Longrightarrow T$ is self-adjoint.

Here is a special case: $V = \mathbb{P}_2$, $[a,b] = [0,1]$, $q = 0$, $p = x(1-x)$. We have $T(1) = 0$, $T(x) = 1-2x$, and $T(x^2) = 4x - 6x^2$. In the basis $\mathcal{A} := \{1, x, x^2\}$,

$$[T]_{\mathcal{A}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 4 \\ 0 & 0 & -6 \end{pmatrix}$$ — nothing special.

But if we apply Gram-Schmidt starting with 1, we get $\{\vec{u}_1 = 1,\ \vec{u}_2 = \sqrt{12}\,(x - \frac{1}{2}),\ \vec{u}_3 = 3\sqrt{10}\,(x^2 - x + \frac{1}{6})\} =: \mathcal{B}$ and

$$[T]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$ — an o.n. eigenbasis, just like the Spectral Theorem told us must exist!

What happens with a more "random" o.n. basis like $\{\vec{w}_1 = \sqrt{5}\,x^2,\ \vec{w}_2 = 4\sqrt{3}\,(x - \frac{5}{4}x^2),\ \vec{w}_3 = 3 - 12x + 10x^2\} =: \mathcal{C}$?

$$[T]_{\mathcal{C}} = \begin{pmatrix} -1 & \sqrt{\frac{5}{3}} & 0 \\ \sqrt{\frac{5}{3}} & -3 & \frac{4}{\sqrt{3}} \\ 0 & \frac{4}{\sqrt{3}} & -4 \end{pmatrix}$$ — a symmetric matrix. Perhaps a general rule? //

Coordinates

Given an operator $T: V \to V$ on a finite-dimensional inner-product space, let $\mathcal{A} = \{\vec{\alpha}_1, \dots, \vec{\alpha}_n\}$ be any basis of V. Put $A := [T]_{\mathcal{A}}$ — whose entries are def'd by $T\vec{\alpha}_j = \sum_i a_{ij}\vec{\alpha}_i$. If $\mathcal{A}$ is o.n. then $\langle T\vec{\alpha}_j, \vec{\alpha}_i \rangle = \langle \sum_k a_{kj}\vec{\alpha}_k, \vec{\alpha}_i \rangle = \sum_k a_{kj} \underbrace{\langle \vec{\alpha}_k, \vec{\alpha}_i \rangle}_{\delta_{ki}} = a_{ij}$ gives a <u>formula</u> for the matrix coefficients.

Let $S: V \to V$ be another operator, and observe that

$$[S]_{\mathcal{A}} = A^* (:= \bar{A}^T) \iff S\vec{\alpha}_j = \sum_i \overline{a_{ji}}\, \vec{\alpha}_i$$

$$\overset{\mathcal{A} \text{ o.n.}}{\iff} \langle S\vec{\alpha}_j, \vec{\alpha}_i \rangle = \overline{a_{ji}}$$

$$\iff \langle \vec{\alpha}_i, S\vec{\alpha}_j \rangle = a_{ji}$$

$$\iff \langle \vec{\alpha}_j, S\vec{\alpha}_i \rangle (= a_{ij}) = \langle T\vec{\alpha}_j, \vec{\alpha}_i \rangle$$

$$\iff \langle \vec{u}, S\vec{v} \rangle = \langle T\vec{u}, \vec{v} \rangle \quad \forall \vec{u}, \vec{v} \in V.$$

Hermitian transpose

(A^T in the real case)

This establishes existence & uniqueness of the adjoint $T^\dagger : V \to V$, and shows that if $[T]_{\mathcal{A}} = A$ then $[T^\dagger]_{\mathcal{A}} = A^*$.
It also proves

<u>Prop. 3</u>: (Still assuming $\mathcal{A}$ is o.n.)

- T is self-adjoint $\iff T = T^\dagger \iff A = A^*$ A Hermitian
- T is skew-adjoint $\iff T = -T^\dagger \iff A = -A^*$ A skew-Hermitian
- T is unitary $\iff T^\dagger T = I \iff A^* A = \mathbb{I}_n \iff A^* = A^{-1}$. A unitary

Notice that in the real case, "unitary" becomes "orthogonal".

Now let $\mathcal{B}$ be an o.n. eigenbasis of T (in one of the 3 cases above). Take $U: V \to V$ to be the unique transformation sending $\vec{\beta}_i \mapsto \vec{\alpha}_i$ for each i. Then <u>U is unitary</u>: on arbitrary $\vec{v} = \sum_i v_i \vec{\beta}_i$, $\vec{w} = \sum_j w_j \vec{\beta}_j$, $\langle U\vec{v}, U\vec{w} \rangle = \sum_i \sum_j v_i \bar{w}_j \langle U\vec{\beta}_i, U\vec{\beta}_j \rangle$

$$= \sum_i \sum_j v_i \bar{w}_j \underbrace{\langle \vec{\alpha}_i, \vec{\alpha}_j \rangle}_{\delta_{ij}} = \sum_i v_i \bar{w}_i = \sum_i \sum_j v_i \bar{w}_j \langle \beta_i, \beta_j \rangle = \langle \vec{v}, \vec{w} \rangle.$$

Moreover, $UTU^{-1}\vec{\alpha}_i = UT\vec{\beta}_i = U\lambda_i \vec{\beta}_i = \lambda_i U\vec{\beta}_i = \lambda_i \vec{\alpha}_i \implies$

$\mathcal{A}$ is an (o.n.) eigenbasis for UTU^{-1} $\Rightarrow$

$$D := \operatorname{diag}\{\lambda_1, \dots, \lambda_n\} = [UTU^{-1}]_{\mathcal{A}} = \overbrace{[U]_{\mathcal{A}}}^{=:P \ \text{unitary} \Rightarrow} [T]_{\mathcal{A}} \overbrace{[U]_{\mathcal{A}}^{-1}}^{= P^{-1} = P^*}$$

$$= P A P^* .$$

The moral:

<u>Prop. 4</u>: If A is the matrix of a self-adjoint, skew-adjoint, or unitary operator T with respect to an o.n. basis, then it can be <u>unitarily diagonalized</u>: there exists a unitary (change-of-basis) matrix P s.t. $PAP^* = D$ is diagonal.

The orthogonal diagonalization done two lectures ago is just the real version of this in the special case where $V = \mathbb{R}^n$ with the dot product and $\mathcal{A} = \mathcal{E}$.