Lecture 16: Operators on inner-product spaces

Recall that an inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ on a real vector space must be:
- symmetric: $\langle x, y \rangle = \langle y, x \rangle$
- linear (in both entries):
  $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
  $\langle cx, y \rangle = c \langle x, y \rangle$
- positive definite: $\langle x, x \rangle > 0$ for $x \neq 0$

If $V$ is a complex vector space, one replaces symmetry by H"{o}rmiit symmetry $\langle x, y \rangle = \overline{\langle y, x \rangle}$ which implies conjugate linearity in the second entry: $\langle x, ce \rangle = \overline{c} \langle x, e \rangle$.

Since $\mathbb{R} = \mathbb{C}$ for real numbers, we can discuss both real and complex at once.

A linear endomorphism $T : V \to V$ on an inner product space is often called an operator.

**Definition 1:** $T$ is **self-adjoint** $\iff \langle Tx, y \rangle = \langle x, Ty \rangle \ (Vx, y \in V)$

skew-adjoint $\iff \langle Tx, y \rangle = -\langle x, Ty \rangle$

unitary $\iff \langle Tx, Ty \rangle = \langle x, y \rangle$

**Remark.**

The terminology comes from the notion of the adjoint $T^\dagger$ of $T$, which is supposed to satisfy $\langle Tx, y \rangle = \langle x, T^\dagger y \rangle \ (Vx, y \in V)$. (We will later see that this exists and is unique in the finite-dimensional case.) The three parts of the Defn. correspond to $T^\dagger = T^\dagger$, $T^\dagger = -T^\dagger$, and $T^\dagger T = I$. (Here $I : V \to V$ denotes the identity transformation.)
Prop. 1: Eigenvalues of a self-adjoint operator are either real or pure imaginary.

Proof: If \( T\mathbf{v} = \lambda \mathbf{v} \), then in the first 2 cases
\[
\lambda \langle \mathbf{v}, \mathbf{v} \rangle = \langle \mathbf{v}, T\mathbf{v} \rangle = \pm \langle \mathbf{v}, \mathbf{v} \rangle = \pm \overline{\lambda} \langle \mathbf{v}, \mathbf{v} \rangle \quad \text{and} \quad \langle \mathbf{v}, \mathbf{v} \rangle \neq 0
\]
\[
\Rightarrow \lambda = \pm \overline{\lambda}.
\]
In the unitary case,
\[
\langle \mathbf{v}, \mathbf{v} \rangle = \langle T\mathbf{v}, T\mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{v} \rangle = 1 \Rightarrow \lambda^2 = 1 \Rightarrow |\lambda| = 1.
\]

Prop. 2: Eigenvectors with distinct eigenvalues are orthogonal, in each of the cases in Prop. 1.

Proof: If \( T\mathbf{v} = \lambda \mathbf{v} \) and \( T\mathbf{w} = \eta \mathbf{w} \), then in the first 2 cases
\[
\lambda \langle \mathbf{v}, \mathbf{w} \rangle = \langle T\mathbf{v}, \mathbf{w} \rangle = \pm \langle \mathbf{v}, T\mathbf{w} \rangle = \pm \overline{\eta} \langle \mathbf{v}, \mathbf{w} \rangle \Rightarrow \lambda = \eta \langle \mathbf{v}, \mathbf{w} \rangle \]
\[
\Rightarrow \langle \mathbf{v}, \mathbf{w} \rangle = 0.
\]
In the unitary case, \( \langle \mathbf{v}, \mathbf{w} \rangle = \langle T\mathbf{v}, T\mathbf{w} \rangle = \overline{\lambda} \langle \mathbf{v}, \mathbf{w} \rangle \) and
\[
\lambda \neq -\overline{\lambda} \Rightarrow \langle \mathbf{v}, \mathbf{w} \rangle = 0.
\]

Example: Let \( V \) be the infinitely differentiable (possibly complex-valued) functions on \( \mathbb{R} \) which are periodic with period 1: \( f \in V \Rightarrow f(x + 1) = f(x) \ \forall x \in \mathbb{R} \). Define an inner product by \( \langle f, g \rangle = \int_0^1 f(x) \overline{g(x)} \, dx \), and note that \( V \) is closed under differentiation — \( T = \frac{d}{dx} : V \to V \).

Then \( \langle Tf, g \rangle = \int_0^1 f'(x) \overline{g(x)} \, dx = f \overline{g}' \big|_0^1 - \int_0^1 f(x) \overline{g'(x)} \, dx = -\langle f, Tg \rangle \Rightarrow T \) is skew-adjoint \( \Rightarrow T^2 \) is self-adjoint (\( \langle T^2 f, g \rangle = \langle Tf, Tg \rangle = \langle f, T^2 g \rangle \)).
Since $\hat{v}_k = \sin \left( \frac{Kx}{2\pi} \right)$ are eigenvectors with distinct eigenvalues $\lambda_k = -\frac{K^2}{4\pi^2}$ under $T^2$, they are automatically orthogonal:
\[
\int_0^1 \sin \left( \frac{Kx}{2\pi} \right) \sin \left( \frac{Lx}{2\pi} \right) dx = 0 \quad \text{if} \quad k \neq l.
\]

Spectral Theorem: Assume that $V$ is finite-dimensional. Then $T$ has an orthonormal eigenbasis.

**Proof:** Let $\lambda$ be any root of $0 = \det (\lambda I - T)$.

Then an eigenvector $v_\lambda$ exists: $Tv_\lambda = \lambda v_\lambda$. Normalize so $\|v_\lambda\| = 1$.

Let $W = \{w \in V \mid \langle w, v_\lambda \rangle = 0\}$. Then $T$ restricts to $W$:
\[
\langle T w, v_\lambda \rangle = \langle w, T v_\lambda \rangle = \langle w, v_\lambda \rangle = 0 = 0 \Rightarrow T \in W.
\]
(In the unity case, $v_\lambda = \bar{v}_{\bar{\lambda}}$, $v_\lambda = \bar{v}_{\bar{\lambda}}$, $T v_\lambda = \lambda \bar{w}$. $\langle T w, v_\lambda \rangle = \langle T w, v_\lambda \rangle = \lambda \bar{w}$, $\langle w, \bar{w} \rangle = 0 = 0 \Rightarrow T \in W$.) Since $\dim W = \dim V - 1$, proof is done by induction on dimension. \(\square\)

**Ex.**

Let $p, q \in C^1 [a, b]$, with $p(a) = p(b) = 0$, and $V \subseteq C^\infty [a, b]$ any subspace closed under the operator $T f := (pf')' + qf + f$; the inner product is $\langle f, g \rangle := \int_a^b f(x) g(x) dx$. We calculate
\[
\langle Tf, g \rangle - \langle f, T g \rangle = \int_a^b \left( (pf')' + qf + f \right) g - ((pf')' + qf + f) g dx
\]
\[
= \int_a^b pf' g dx - \left[ pf g \right]^b_a - \int_a^b pf' g dx - \left( (pg')' - \int_a^b pg' dx \right)
\]
\[
= 0 \Rightarrow T \text{ is self-adjoint.}
\]
Here is a special case: \( V = P_2 \), \([a, b] = [0, 1]\), \( q = 0 \), \( p = x(1 - x) \). We have \( T(1) = 0, T(x) = 1 - 2x \), and \( T(x^2) = 4x - 6x^2 \). In the basis \( A = \{1, x, x^2\} \),

\[
[T]_A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 4 \\ 0 & 0 & -6 \end{pmatrix} \quad \text{— nothing special.}
\]

But if we apply Gram-Schmidt starting with 1, we get \( \vec{e}_1 = 1 \), \( \vec{e}_2 = \sqrt{2} (x - \frac{1}{2}) \), \( \vec{e}_3 = 3\sqrt{10} (x^2 - x + \frac{1}{2}) \) \( \vec{e}_4 = : B \) and

\[
[T]_B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -6 \end{pmatrix} \quad \text{— an o.n. eigenbasis, just like the Spectral Theorem told us must exist!}
\]

What happens with a more "random" o.n. basis like \( \{\vec{w}_1 = \sqrt{3} x, \vec{w}_2 = 4 \sqrt{5} (x - \frac{3}{4} x^2), \vec{w}_3 = 3 - 12x + 10 x^4\} =: C \) ?

\[
[T]_C = \begin{pmatrix} -1 & 0 & 0 \\ \sqrt{3} & -3 & 0 \\ 0 & \frac{4}{\sqrt{3}} & -4 \end{pmatrix} \quad \text{— a symmetric matrix. Perhaps a general rule?}
\]

Coordinates

Given an operator \( T: V \to V \) on a finite-dimensional inner-product space, let \( A = \{\vec{a}_1, ..., \vec{a}_n\} \) be any basis \& \( V \). Put \( A := [T]_A \) — whose entries are defined by \( T \vec{a}_j = \sum \frac{\sqrt{\lambda}}{i} \vec{a}_i \). If \( A \) is o.n. then \( \langle T \vec{a}_j, \vec{a}_i \rangle = \langle \sum \frac{\sqrt{\lambda}}{i} \vec{a}_k, \vec{a}_i \rangle = \sum \frac{\sqrt{\lambda}}{i} \langle \vec{a}_k, \vec{a}_i \rangle = \frac{\sqrt{\lambda}}{i} \delta_{kj} \delta_{ki} \) gives a formula for the matrix coefficients.
Let $S : V \to V$ be another operator, and observe that

\[
[S]_A = A^* (:= A^T) \iff S_{ij} = \sum_k a_{kj} \tilde{x}_i \tilde{x}_j
\]

\[
\iff \langle S \tilde{x}_i, \tilde{x}_j \rangle = \overline{a_{ij}}
\]

\[
\iff \langle \tilde{x}_i, S \tilde{x}_j \rangle = a_{ij}
\]

\[
\iff \langle \tilde{x}_i, S \tilde{x}_j \rangle (\in \mathbb{C}) = \langle T \tilde{x}_i, \tilde{x}_j \rangle
\]

\[
\iff \langle \tilde{x}_i, S \tilde{x}_j \rangle \in \mathbb{C} \quad \forall \tilde{x}_i, \tilde{x}_j \in V.
\]

This establishes existence & uniqueness of the adjoint $T^* : V \to V$, and shows that if $[T]_A = A$ then $[T^*]_A = A^*$.

It also gives

Prop. 3: (Still assuming $A$ is o.n.)

- $T$ is self-adjoint $\iff T = T^* \iff A = A^*$ A Hermitian
- $T$ is skew-adjoint $\iff T = -T^* \iff A = -A^*$ A skew-Hermitian
- $T$ is unitary $\iff T^* T = I \iff A^* A = I_n \iff A^* = A^{-1}$. A unitary

Notice that in the real case, "unitary" becomes "orthogonal".

Now let $B$ be an o.n. eigenspace of $T$ (in one of the

3 cases above). Take $U : V \to V$ to be the unique transformation sending $\tilde{\beta}_i \mapsto \tilde{x}_i$ for each $i$. Then $U$ is unitary: on arbitrary

\[
\tilde{v} = \sum_i \tilde{\beta}_i, \quad \tilde{w} = \sum_j \tilde{\beta}_j,
\]

\[
\langle U \tilde{v}, U \tilde{w} \rangle = \sum_{ij} \tilde{v}_i \overline{\tilde{w}_j} \sum_i \overline{\tilde{v}_i} \tilde{w}_i = \sum_i \overline{\tilde{v}_i} \tilde{v}_i \sum_j \overline{\tilde{w}_j} \tilde{w}_j = \langle \tilde{v}, \tilde{w} \rangle.
\]

Moreover, $U^* U \tilde{z}_i = U^* U \tilde{\beta}_i = U^* U \lambda_i \tilde{\beta}_i = \lambda_i U \tilde{\beta}_i = \lambda_i \tilde{z}_i \iff$
Prop. 4: If $A$ is the matrix of a self-adjoint, skew-adjoint, or unitary operator $T$ with respect to an o.n. basis, then it can be unitarily diagonalized: there exists a unitary (change-of-basis) matrix $P$ s.t. $PAP^* = D$ is diagonal.

The orthogonal diagonalization done two lectures ago is just the real version of this in the special case where $V = \mathbb{R}^n$ with the dot product and $A = E$. 