Lecture 17: Homogeneous Linear DEs

Recall from last Fall how we solve a homogeneous linear DE of 2nd order with constant coefficients

$$f''(x) + af'(x) + bf(x) = 0. \qquad (a, b \in \mathbb{R})$$

First pass to the characteristic equation

$$0 = r^2 + ar + b = (r - r_1)(r - r_2),$$

then write $f(x) = \begin{cases} c_1 e^{r_1 x} + c_2 e^{r_2 x} & \text{if } r_1 \neq r_2 \ * \\ c_1 e^{rx} + c_2 x e^{rx} & \text{if } r_1 = r_2 =: r. \end{cases}$

The point is that <u>the solutions form a vector space V of dimension 2</u>. Notice that the roots of the characteristic equation are the eigenvalues of $\frac{d}{dx} : V \to V$ — i.e. it is the characteristic polynomial of $\frac{d}{dx}$ (hence the name). Moreover, we may view V as the kernel of an operator: the linear transformation $L : C^2(\mathbb{R}) \to C^0(\mathbb{R})$ given by $L = D^2 + aD + b$ (recall $D = \frac{d}{dx}$).

<u>How might this generalise to higher order?</u>

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* If r_1, r_2 are complex, then $r_1 = A + iB$ & $r_2 = \bar{r}_1 = A - iB$; to get real solutions, take $\frac{e^{r_1 x} + e^{r_2 x}}{2} = e^{Ax} \cos Bx$ & $\frac{e^{r_1 x} - e^{r_2 x}}{2i} = e^{Ax} \sin Bx$.

Let $I \subset \mathbb{R}$ be an open interval, and $x_0 \in I$.

For a linear differential operator $L = D^n + P_1 D^{n-1} + \cdots + P_n$ of order n, with $P_j \in C^0(I)$, we will prove next week:

<u>Theorem</u>: Given any vector $\vec{y} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix} \in \mathbb{R}^n$, there exists a unique solution f to $Lf = 0$ with $f(x_0) = y_0$, $f'(x_0) = y_1, \ldots,$ $f^{(n-1)}(x_0) = y_{n-1}$.

Defining $T : C^n(I) \to \mathbb{R}^n$ by $T(f) = \begin{pmatrix} f(x_0) \\ f'(x_0) \\ \vdots \\ f^{(n-1)}(x_0) \end{pmatrix}$, we can restate the initial-value condition as $Tf = \vec{y}$. Viewing $L : C^n(I) \to C^0(I)$ as a transformation, the solution space is $V := \ker(L)$.

The restriction of T to V

$$T|_V : V \to \mathbb{R}^n$$

is onto (by the existence part of the Theorem) and 1-1 (by the uniqueness part, which gives $Tf = \vec{0} \Rightarrow f \equiv 0$), hence an isomorphism. It follows that $\dim V = \dim \mathbb{R}^n = n$ and that there exists a linearly independent set $\{u_1, \ldots, u_n\}$ of solutions spanning V.

<u>Here is how to find these $\{u_j\}$ in a special case.</u>

Consider $L = \sum_{j=0}^{n} a_j D^{n-j}$ an operator with <u>constant real coeffs.</u>

Its <u>characteristic polynomial</u> is defined to be $p_L(r) := \sum_{j=0}^{n} a_j r^{n-j}$.

If L_1 & L_2 are 2 such operators, then $L_1 L_2 = L_2 L_1$

and $p_{L_1 L_2} = p_{L_1} p_{L_2}$ (& $p_{a_1 L_1 + a_2 L_2} = a_1 p_{L_1} + a_2 p_{L_2}$).

Now if $p_L = a_0 (r - r_1) \cdots (r - r_n)$, then

$$L = a_0 (D - r_1) \cdots (D - r_n) =: L_1 \cdots L_n ,$$

and $\ker(L_j) \subset \ker(L) =: V$ for each j since

$$L_j f = 0 \Rightarrow (\underbrace{L_1 \cdots L_j \cdots L_n}_{\text{commute to front}}) f = (L_1 \cdots L_{j-1} L_{j+1} \cdots L_n) \underbrace{L_j f}_{0} = 0.$$

Hence $(D - r_j) e^{r_j x} = 0 \Rightarrow e^{r_j x} \in V$ for each j. If the $\{r_j\}$ are distinct then these n functions are independent* and furnish our $\{u_j\}$.

But what if a root r is repeated m times?

Then we can write $L = L_0 \cdot (D - r)^m$, and $e^{rx}, xe^{rx}, \ldots, x^{m-1} e^{rx}$ are independent and in $\ker((D-r)^m)$ hence V.

(So now you know how to find a basis of the complex solutions. To go from there to a <u>real</u> basis, just observe that if r_j is complex, then some other $r_i = \overline{r_j}$. Then

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* See the inductive proof on p. 1 of lecture 48 from last term.

$\frac{e^{a_j x} + e^{\bar{\lambda}_j x}}{2} = \cos(\lambda_j x)$ and $\frac{e^{\lambda_j x} - e^{\bar{\lambda}_j x}}{2i} = \sin(\lambda_j x)$ are real

and replace the pair $e^{a_j x}, e^{\bar{\lambda}_j x}$ in the complex basis.

If one has repeated complex roots (m times) then you need

$\cos(\lambda_j x), \sin(\lambda_j x), x\cos(\lambda_j x), x\sin(\lambda_j x), \ldots, x^{m-1}\cos(\lambda_j x), x^{m-1}\sin(\lambda_j x))$.

Ex/

$$L = D^5 - 8D^3 + 3D^2 + 4D + 12$$

$$= (D-2)^2 (D-3) \underbrace{(D^2 + D + 1)}_{\left(D - \frac{-1+\sqrt{3}i}{2}\right)\left(D - \frac{-1-\sqrt{3}i}{2}\right)}$$

$$\Rightarrow \quad V = \text{span}\left\{ e^{2x}, xe^{2x}, e^{3x}, e^{-\frac{1}{2}x}\cos\left(\frac{\sqrt{3}}{2}x\right), e^{-\frac{1}{2}x}\sin\left(\frac{\sqrt{3}}{2}x\right) \right\}.$$

That's it!

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