Lecture 18: Inhomogeneous linear DEs

Let $L = D^n + P_1(x)D^{n-1} + P_2(x)D^{n-2} + \dots + P_n(x)$ $(P_j \in C^0(I))$

be a linear differential operator of order n. Recall that, viewing L as a linear transformation from $C^n(I) \to C^0(I)$ ($I = (a,b)$ an open interval), the solution space

$$\ker(L) = \{ f \in C^n(I) \mid Lf = 0 \}$$

(i.e. f satisfying $f^{(n)} + P_1 f^{(n-1)} + \dots + P_n f = 0$)

has dimension n, with basis $\{u_1, \dots, u_n\}$.

Given $R(x) \in C^0(I)$, we are interested in the set $\underline{\underline{S}}$ of solutions to the inhomogeneous equation

$$(*) \qquad \boxed{LF = R}.$$

Since any two solutions to this differ by a solution of the homogeneous equation, if F is one solution then

$$S = F + \ker(L) = \left\{ F + \sum_{j=1}^{n} c_j u_j \;\middle|\; c_j \in \mathbb{R} \right\}.$$

(This is a translate of the vector space $\ker(L)$, but is not a vector space itself.)

KEY QUESTION: How do we find an F?

Apostol calls this "$y_1(x)$"

I will now suggest three approaches.

Approach #1: ANNIHILATOR METHOD

This works only if L has constant coefficients AND there is an operator M with constant coefficients which annihilates $R(x)$: $MR = 0$.

Applying M to both sides of $(*)$ gives

$$(ML)F = MR = 0$$

$\Rightarrow F \in \ker(ML)$. For simplicity assume the characteristic polynomials p_M & p_R have distinct roots, so that $\ker(ML) = \ker(M) + \ker(L)$. Obviously we can subtract out the $\ker(L)$ part, so that $F \in \ker(M)$.

Will any solution of $MF = 0$ do? No, but some solution must: so if $\omega_1, \dots, \omega_m$ is a basis of $\ker(M)$ we write $L(\sum a_j \omega_j) = R$ and solve for $\vec{a}$.

Ex 1/ Solve $F'' - 5F' + 6F = xe^x$, i.e.

$LF = R$ where $L = D^2 - 5D + 6 = (D-3)(D-2)$ & $R = xe^x$.

Take $M = (D-1)^2$, $\{\omega_1 = e^x, \omega_2 = xe^x\}$, and solve

$xe^x = L(a_1 e^x + a_2 x e^x) = (2a_1 - 3a_2)e^x + (2a_2)xe^x \Rightarrow a_2 = \frac{1}{2}, a_1 = \frac{3}{4}$

$\Rightarrow F = \frac{3}{4}e^x + \frac{1}{2}xe^x$, and general solution to $Lf = R$ is

$f = F + c_1 e^{2x} + c_2 e^{3x}$. //

Approach #2: REDUCTION TO FIRST-ORDER

it can handle some R's that Approach #1 can't (see e.g. Apostol p. 163)

Again, this works when L has constant coefficients.

To solve $LF = R$, write $L = (D - r)L_0$ ← root of P_L

and $G = L_0 F$. The equation becomes

$$(D-r)G = R \quad \rightsquigarrow \quad G' - rG = R.$$

Last semester, we proved that $y' + P(x)y = R(x)$ has the "particular solution" $y(x) = e^{-A(x)}\int_{x_0}^{x} R(t)e^{A(t)}dt$ where $A(x) := \int_{x_0}^{x} P(u)\,du$ and $x_0 \in I$. Here we can apply this with $P = -r \Rightarrow A = r(x_0 - x)$ to get

$$G = e^{r(x-x_0)}\int_{x_0}^{x} R(t)e^{r(x_0-t)}dt.$$

Now we have reduced the problem to solving

$$L_0 F = G,$$

with G now playing the rôle of R. Keep going in this fashion.

Ex 2/ Same eqn. as Ex. 1: solve $(D-3)\underbrace{(D-2)F}_{=:G} = xe^x$. (take $x_0 = 0$)

$(D-3)G = xe^x$ has sol'n. $G = e^{3x}\int_0^x te^te^{-3t}dt = e^{3x}\int_0^x te^{-2t}dt$

$\overset{\int \text{by}}{\underset{\text{parts}}{=}} e^{3x}\left(-\frac{t}{2}e^{-2t}\Big|_0^x + \frac{1}{2}\int_0^x e^{-2t}dt\right) = -\frac{x}{2}e^x - \frac{1}{4}e^x + \frac{1}{4}e^{3x}$. Now solve

$(D-2)F = G$: $\quad F = e^{2x}\int_0^x G(t)e^{-2t}dt = \frac{3}{4}e^x + \frac{1}{2}xe^x + \frac{1}{4}e^{3x} - e^{2x}$,

where the last two terms are in $\ker(L)$ so can be ignored. //

Approach #3: VARIATION OF PARAMETERS — no constant-coeffs requirement

Let $\vec{u} : I \to \mathbb{R}^n$ be the vector-valued function $\begin{pmatrix} u_1(x) \\ \vdots \\ u_n(x) \end{pmatrix}$, so $L\vec{u} = \vec{0}$. We want to pick $\vec{v} : I \to \mathbb{R}^n$ so that $F = \vec{v} \cdot \vec{u}$ solves $LF = R$. (Here the "varying parameters" are the components $\{v_i\}$ of $\vec{v}$.) In fact, if

$\vec{v}\,' \cdot \vec{u}^{(k)} = 0$ for $k = 0, 1, \ldots, n-2$ and $\vec{v}\,' \cdot \vec{u}^{(n-1)} = R$,

then $F' = \vec{v} \cdot \vec{u}\,'$, $F'' = \vec{v} \cdot \vec{u}\,''$, $\ldots$, $F^{(n-1)} = \vec{v} \cdot \vec{u}^{(n-1)}$,

and $F^{(n)} = R + \vec{v} \cdot \vec{u}^{(n)}$ $\Longrightarrow$

$$LF = F^{(n)} + P_1(x) F^{(n-1)} + \cdots + P_n(x) F = R + \vec{v} \cdot \overbrace{\left(\vec{u}^{(n)} + P_1 \vec{u}^{(n-1)} + \cdots + P_n \vec{u}\right)}^{=0}$$

$$= R.$$

The conditions on $\vec{v}\,'$ read

$$\begin{pmatrix} \leftarrow \vec{u} \rightarrow \\ \leftarrow \vec{u}\,' \rightarrow \\ \vdots \\ \leftarrow \vec{u}^{(n-1)} \rightarrow \end{pmatrix} \begin{pmatrix} v_1' \\ \vdots \\ v_n' \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ R \end{pmatrix},$$

Wronskian $W(x)$

i.e. $W \vec{v}\,' = R \vec{e}_n$.

<u>Claim</u>: $W(x)$ is invertible for all $x \in I$.

<u>Proof</u>: If $w(x)$ is zero at some $a \in I$, then $W(a)$ is $\underbrace{\text{not invertible}}_{\text{a.k.a. "singular"}}$, hence $\exists\, \vec{c} \neq \vec{0}$ s.t. $W(a)\vec{c} = \vec{0}$. Writing $f := \vec{c} \cdot \vec{u}$, this says that

$$\boxed{f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0} \quad (\dagger).$$

But $Lf = 0 \Rightarrow f$ is the unique solution satisfying $(\dagger)$

$$\Rightarrow f = 0 \xRightarrow[\{u_i\}\text{ indep.}]{} \vec{c} = \vec{0}.$$

So $W(x)$ is nowhere zero, and $\therefore$ $W(x)$ is invertible for all $x \in I$. $\square$

Thus we may write $\vec{v}' = R W^{-1} \vec{e}_n \Longrightarrow$

$$\vec{v} = \int_{x_0}^{x} R(t)\, W(t)^{-1} \vec{e}_n \, dt.$$

(As stated above, our solution is then $F = \vec{v} \cdot \vec{u}$.)

Ex 3 / Same eqn. as Ex's 1-2. $W = \begin{pmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{pmatrix}$

$$\Rightarrow W^{-1} = e^{-5x} \begin{pmatrix} 3e^{3x} & -e^{3x} \\ -2e^{2x} & e^{2x} \end{pmatrix} \Rightarrow W(t)^{-1} \vec{e}_{\underset{2}{n}} = \begin{pmatrix} -e^{-2x} \\ e^{-3x} \end{pmatrix}$$

$$\Rightarrow \vec{v} = \int_0^x \begin{pmatrix} -te^{-t} \\ te^{-2t} \end{pmatrix} dt = \begin{pmatrix} xe^{-x} + e^{-x} - 1 \\ -\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x} + \frac{1}{4} \end{pmatrix}.$$

We can drop the constants since $\overrightarrow{\text{const}} \cdot \vec{u} \in \ker(L)$. So

$$F = \vec{v} \cdot \vec{u} = \left(xe^{-x} + e^{-x}\right) e^{2x} + \left(-\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x}\right) e^{3x}$$

$$= \frac{3}{4} e^{x} + \frac{1}{2} x e^{x}. \quad //$$

Appendix: A formula for the Wronskian

Consider $w(x) := \det(W(x)) = \begin{vmatrix} \leftarrow \vec{u} \rightarrow \\ \leftarrow \vec{u}' \rightarrow \\ \vdots \\ \leftarrow \vec{u}^{(n-1)} \rightarrow \end{vmatrix}$, and

recall from HW that $w'(x) = \begin{vmatrix} \leftarrow \vec{u} \rightarrow \\ \leftarrow \vec{u}' \rightarrow \\ \vdots \\ \leftarrow \vec{u}^{(n-2)} \rightarrow \\ \leftarrow \vec{u}^{(n)} \rightarrow \end{vmatrix}$ (using "Leibniz rule" on the rows and the fact that 2 rows $= \Rightarrow \det = 0$).

By linearity of det in the last row,

$$w' + P_1 w = \begin{vmatrix} \leftarrow \vec{u} \rightarrow \\ \leftarrow \vec{u}' \rightarrow \\ \vdots \\ \leftarrow \vec{u}^{(n-2)} \rightarrow \\ \leftarrow \underbrace{\vec{u}^{(n)} + P_1 \vec{u}^{(n-1)}}_{= -\sum_{j=2}^{n} P_j \vec{u}^{(n-j)}} \rightarrow \end{vmatrix} = 0$$

$-\sum_{j=2}^{n} P_j \vec{u}^{(n-j)} =$ linear comb. of first $n-1$ rows

$$\Rightarrow \quad w' = -P_1 w \quad \Rightarrow \quad w(x) = w(x_0)\, e^{-\int_{x_0}^{x} P_1(t)\, dt}.$$