# Lecture 19: Power series solutions

So far, we have learned a great deal about solving equations of the form $Lf = 0$ if $L$ has constant coefficients, and solving $LF = R$ in general once one has a basis of solutions to $Lf = 0$. But what about solving $Lf = 0$ if $L$ doesn't have constant coefficients?

The problem is that the solutions may not be expressible in terms of familiar functions. (In fact one frequently <u>uses</u> DEs to define <u>new transcendental functions</u>, like the Bessel, hypergeometric, Legendre, & Airy functions.)

One natural idea is to try to solve the DE in terms of power series, and provided the coefficient functions of $L$ are analytic — i.e. expressible in terms of (convergent) power series — this works well.

So let $I = (x_0 - r, x_0 + r)$, and say that

$$\underbrace{f \in \mathcal{A}(I)}_{\text{analytic functions on } I} \overset{\text{defn.}}{\iff} f(x) = \sum_{n \geq 0} a_n (x - x_0)^n \text{ on } I.$$

We'll assume $L = D^2 + P_1(x) D + P_2(x)$, with $P_1, P_2 \in \mathcal{A}(I)$; the only reason to restrict to operators of order 2 is to keep the notation from getting too messy.

**Theorem**: The solution space $\ker(L : \mathcal{A}(I) \to \mathcal{A}(I))$ has dimension 2. (More generally, if $L$ has order $n$, the dimension is $n$, though I won't prove that.)

**Proof**: Wolog $x_0 = 0$. Write $P_1 = \sum_{n\geq 0} b_n x^n$, $P_2 = \sum_{n\geq 0} c_n x^n$, $f = \sum_{n\geq 0} a_n x^n$. We must solve $Lf = 0$ for $\{a_n\}$.

- $P_2 f = \sum_{n\geq 0} \left( \sum_{k=0}^{n} a_k c_{n-k} \right) x^n$
- $P_1 f' = P_1 \sum_{n\geq 1} n a_n x^{n-1} = P_1 \sum_{n\geq 0} (n+1) a_{n+1} x^n = \sum_{n\geq 0} \left( \sum_{k=0}^{n} (k+1) a_{k+1} b_{n-k} \right) x^n$
- $f'' = \sum_{n\geq 2} n(n-1) a_n x^{n-2} = \sum_{n\geq 0} (n+2)(n+1) a_{n+2} x^n$

$$\Rightarrow Lf = \sum_{n\geq 0} \left\{ (n+2)(n+1) a_{n+2} + \sum_{k=0}^{n} a_k c_{n-k} + \sum_{k=1}^{n+1} k a_k b_{n-k+1} \right\} x^n$$

$$\underset{Lf=0}{\Longrightarrow} \quad a_{n+2} = - \frac{\sum_{k=0}^{n} a_k c_{n-k} + \sum_{k=1}^{n+1} k a_k b_{n-k+1}}{(n+1)(n+2)}$$

defines $a_{n+2}$ in terms of $a_0, a_1, a_2, \ldots, a_{n+1}$

$\Rightarrow \{a_n\}$ is determined by $a_0, a_1$ which may be chosen freely & yield 2 independent solutions
$u_1 = 1 + 0x + \text{higher-order terms}$
$u_2 = 0 + 1x + \text{〃 〃 〃}$

provided the resulting power series converge on $I$.

To check that $\sum a_n |x|^n$ converges for $x \in I = (-r,r)$, pick any $t \in (0,r)$. The radius of convergence of $P_1$ & $P_2$ are $\geq r$, so $|b_k| \leq \frac{M}{t^k}$, $|c_k| \leq \frac{M}{t^{k+1}}$ for some fixed $M$

$$\Rightarrow (n+1)(n+2)|a_{n+2}| \leq \sum_k k|a_k||b_{n-k+1}| + \sum_k |a_k||c_{n-k}|$$
$$\leq \sum k|a_k| \frac{M}{t^{n-k+1}} + \sum |a_k| \frac{M}{t^{n-k+1}}$$
$$\leq \left(\sum (k+1)|a_k| t^k\right) \frac{M}{t^{n+1}}.$$

Let $A_0 = |a_0|$, $A_1 = |a_1|$, & define

$$A_{n+2} = \frac{M}{(n+1)(n+2)t^{n+1}} \sum_{k=0}^{n+1} (k+1) A_k t^k.$$ Then $|a_n| \leq A_n$ $(\forall n)$

$\Rightarrow \sum a_n t^n$ converges if $\sum A_n t^n$ does.

But $$(n+1)(n+2) A_{n+2} = \frac{M}{t^{n+1}} \sum_{k=0}^{n+1} (k+1) A_k t^k$$
$$-\left[\frac{n(n+1)A_{n+1}}{t} = \frac{M}{t^{n+1}} \sum_{k=0}^{n} (k+1) A_k t^k\right]$$

gives $$(n+1)(n+2) A_{n+2} = \frac{n(n+1)A_{n+1}}{t} + \frac{M}{\cancel{t^{n+1}}}(n+2)A_{n+1}\cancel{t^{n+1}}$$

$$\Rightarrow A_{n+2} = \frac{n(n+1) + M(n+2)t}{(n+1)(n+2)\,t} A_{n+1}.$$

So $$\lim_{n\to\infty} \frac{A_{n+2}|x|^{n+2}}{A_{n+1}|x|^{n+1}} = \lim_{n\to\infty} \frac{n(n+1) + M(n+2)t}{(n+1)(n+2)\,t}|x| = \frac{|x|}{t}$$

is less than 1 (and $\sum A_n |x|^n$ converges) for $|x| < t$. Since $t \in (0,r)$ was arbitrary, $\sum A_n |x|^n$ converges for $|x| < r$. $\square$

## An example

$$L = D^2 - \frac{2x}{1-x^2} D + \frac{\alpha(\alpha+1)}{1-x^2}, \qquad I = (-1,1).$$

$P_1 = \frac{2x}{1-x^2}$, $P_2 = \frac{\alpha(\alpha+1)}{1-x^2}$, $\alpha \in \mathbb{R}$

Solutions to $Lf = 0$ are same as those of $\tilde{L} f = 0$ when $\tilde{L} = (1-x^2)D^2 - 2xD + \alpha(\alpha+1)$. They are called Legendre functions. Writing $f = \sum a_n x^n$,

$$\begin{aligned} 0 &= (1-x^2) D^2 \sum a_n x^n - 2x D \sum a_n x^n + \alpha(\alpha+1) \sum a_n x^n \\ &= \sum_n \left\{ \left( (n+1)(n+2) a_{n+2} - n(n-1) a_n \right) - 2n a_n + \alpha(\alpha+1) a_n \right\} x^n \end{aligned}$$

$$\Rightarrow \quad a_{n+2} = \frac{(n-\alpha)(n+\alpha+1)}{(n+1)(n+2)} a_n. \quad \text{So if you start with}$$

- $a_0 = 1$, $a_1 = 0$, you get an even function $u_1$ — and if $\alpha$ is an even integer $(\geq 0)$ $n-\alpha$ is zero when $n = \alpha$ with the consequence that $a_n = 0$ for $n > \alpha$ $\Rightarrow$ $u_1$ is a polynomial.
- $a_0 = 0$, $a_1 = 1$, you get an odd function $u_2$ — and if $\alpha$ is an odd integer $(\geq 0)$ then $a_n = 0$ for $n > \alpha$ $\Rightarrow$ $u_2$ is a polynomial.

These polynomial solutions for $\alpha = m \in \mathbb{N}$ are the Legendre polynomials

$$P_m(x) = \frac{1}{2^m} \sum_{r=0}^{[\frac{m}{2}]} \frac{(-1)^r (2m-2r)!}{r!\,(m-r)!\,(m-2r)!} x^{m-2r} = \frac{1}{2^m m!} \frac{d^m}{dx^m} (x^2-1)^m.$$

In your HW, you'll check directly that these satisfy $LP_m = 0$ but in a slightly different form: notice that $Lf = 0 \Leftrightarrow \tilde{L} f = 0 \Leftrightarrow ((1-x^2) f')' + \alpha(\alpha+1) f = 0$

$$\iff \underbrace{((x^2-1)f')'}_{Tf} = \alpha(\alpha+1)f.$$

(So, you'll check that $TP_m = m(m+1)P_m$.)

In the 2nd example of Lecture 16, we showed that $T$ was self-adjoint in the inner product $\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$, with the consequence that eigenvectors of $T$ with distinct eigenvalues are orthogonal in this inner product. In particular, the $\{P_m(x)\}$ are.