Lecture 19: Power series solutions

So far, we have learned a great deal about solving equations of the form $Lf = 0$ if $L$ has constant coefficients, and solving $LF = R$ in general once we have a basis of solutions to $Lf = 0$. But what about solving $Lf = 0$ if $L$ doesn't have constant coefficients?

The problem is that the solutions may not be expressible in terms of familiar functions. (In fact one frequently uses DEs to define new transcendental functions, like the Bessel, hypergeometric, Legendre, & Airy functions.)

One natural idea is to try to solve the DE in terms of power series, and provided the coefficient functions of $L$ are analytic — i.e. expressible in terms of (convergent) power series — this works well.

So let $I = (x_0 - r, x_0 + r)$, and say that $f \in \mathcal{A}(I) \iff f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ on $I$.

We'll assume $L = D^2 + P_1(x) D + P_2(x)$, with $P_1, P_2 \in \mathcal{A}(I)$; the only reason to restrict to operators...
of order 2 is to keep the notation from getting too messy.

**Theorem:** The solution space \( \ker (L : A(I) \to A(I)) \) has dimension 2. (More generally, if \( L \) has order \( n \), the dimension is \( n \), though I won’t prove that.)

**Proof:** Wolog \( k_0 = 0 \). Write \( P_1 = \sum b_n x^n \), \( P_2 = \sum c_n x^n \), \( f = \sum a_n x^n \). We must solve \( Lf = 0 \) for \( \{a_n\} \).

- \( P_2 f = \sum_{n \geq 0} \left( \sum_{k=0}^n a_k c_{n-k} \right) x^n \)
- \( P_1 f' = P_1 \sum_{n \geq 1} n a_n x^{n-1} = P_1 \sum_{n \geq 0} (n+1) a_{n+1} x^n = \sum_{n \geq 0} \left( \sum_{k=0}^n (n+1) a_{k+1} b_{n-k} \right) x^n \)
- \( f'' = \sum_{n \geq 2} n(n-1) a_n x^{n-2} = \sum_{n \geq 0} (n+2)(n+1) a_{n+2} x^n \)

\[ \implies Lf = \sum_{n \geq 0} \left\{ (n+2)(n+1) a_{n+2} + \sum_{k=0}^n a_k c_{n-k} + \sum_{k=0}^{n+1} k a_k b_{n-k+1} \right\} x^n \]

\[ \implies a_{n+2} = -\frac{\sum_{k=0}^n a_k c_{n-k} + \sum_{k=0}^{n+1} k a_k b_{n-k+1}}{(n+1)(n+2)} \]

Define \( a_{n+2} \) in terms of \( a_0, a_1, a_2, \ldots, a_n \).

\[ \implies \{a_n\} \text{ is determined by } a_0, a_1 \text{ which may be chosen freely to yield 2 independent solutions } < a_1 = 1 + 0x + \text{ higher-order terms} < a_2 = 0 + 1x + \cdots \]

provided the resulting power series converge on \( I \).
To check that $\sum a_n |x|^n$ converges for $x \in I = (-r, r)$, pick any $x \in (0, r)$. The radius of convergence of $P_t$ & $P_{-t}$ are $R$, so $|b_k| \leq \frac{M}{t^{k+1}}$, $|c_k| \leq \frac{M}{t^{k+1}}$ for some fixed $M$

$$\Rightarrow \sum |a_n|^n \leq \sum |a_n||b_{n+1}| + \sum |a_n||c_{n-1}|$$

$$\leq \sum |a_n| \frac{M}{t^{n+1}} + \sum |a_n| \frac{M}{t^{n+1}}$$

$$\leq \left( \sum (k+1)|a_k| t^k \right) \frac{M}{t^{n+1}}.$$

Let $A_0 = |a_0|$, $A_t = |a_1|$, & define

$$A_{n+2} = \frac{M}{(n+1)(n+2)} \sum_{k=0}^{n+1} (k+1)A_k t^k. \text{ Then } |a_{n+1}| \leq A_{n+1} (V_n)$$

$$\Rightarrow \sum a_n t^n \text{ converges if } \sum A_n t^n \text{ does.}$$

$$\text{But } (n+1)(n+2) A_{n+2} = \frac{M}{t^{n+1}} \sum_{k=0}^{n+1} (k+1)A_k t^k$$

$$- \left[ n \frac{(n+1) A_{n+1}}{t} = \frac{M}{t^{n+1}} \sum_{k=0}^{n+1} (k+1)A_k t^k \right]$$

$$\text{giving } (n+1)(n+2) A_{n+2} = n(n+1) A_{n+1} + \frac{M}{t^{n+1}(n+2)} A_{n+1} t^{n+1}$$

$$\Rightarrow A_{n+2} = \frac{n(n+1) + M(n+2)}{(n+1)(n+2)} t A_{n+1}.$$

$$\text{So } \lim_{n \to \infty} \frac{A_{n+2}}{A_{n+1}} = \lim_{n \to \infty} \frac{n(n+1) + M(n+2)}{(n+1)(n+2)} = \frac{|x|}{t}$$

is less than 1 (and $\sum a_n |x|^n$ converges) for $|x| < t$. Since $r \in (0, r)$ was arbitrary, $\sum a_n |x|^n$ converges for $|x| < r$. \qed
An example:

\[ L = D^2 - \frac{2x}{1-x^2} D + \frac{\alpha (x+1)}{1-x^2}, \quad \alpha \in \mathbb{R} \]

Solutions to \( Lf = 0 \) are same as those of \( \tilde{L} f = 0 \) when

\[ \tilde{L} = (-x^2) D^2 - 2x D + \alpha (x+1) \]

They are called **Legendre functions.** Writing \( f = \sum a_n x^n \),

\[ 0 = (-x^2) D^2 \sum a_n x^n - 2x D \sum a_n x^n + \alpha (x+1) \sum a_n x^n \]

\[ = \sum \left\{ \frac{(n+1)(n+2) a_{n+2} - n(n-1) a_n + 2n a_n + \alpha (x+1) a_n}{x^n} \right\} \]

\[ \Rightarrow a_{n+2} = \frac{(n-a)(n+1)}{(n+1)(n+2)} a_n. \quad \text{So if you start with} \]

- \( a_0 = 1, \quad a_1 = 0, \quad \text{you get an even function } u_1 - \text{ and if} \)
  \( \alpha \) is an even integer, \( n-1 \) is zero when \( n = \alpha \) with the
  consequence that \( a_n = 0 \) for \( n > \alpha \Rightarrow u_1 \) is a polynomial.
- \( a_0 = 0, \quad a_1 = 1, \quad \text{you get an odd function } u_2 - \text{ and} \)
  \( \text{if } \alpha \) is an odd integer, \( \text{then } a_n = 0 \) for \( n > \alpha \Rightarrow u_2 \) is a polynomial.

These polynomial solutions for \( \alpha = m \in \mathbb{N} \) are the *Legendre polynomials*:

\[ P_m(x) = \sum_{r=0}^{\lfloor m/2 \rfloor} \frac{(-1)^r (2m-2r)!}{2^m r! (m-r)! (m-2r)!} x^{m-2r} = \frac{d^m}{dx^m} \left( x^2 - 1 \right)^m. \]

In your HW, you'll check directly that these satisfy

\[ L P_m = 0 \]

but in a slightly different form: assume that \( Lf = 0 \) \( \Rightarrow \int \tilde{L} f = 0 \) \( \Rightarrow (1-x^2) f' + \alpha (x+1) f = 0 \).
\[(\Rightarrow) \quad (x^2 \cdot (f'))' = \alpha (\alpha + 1) f.\]

\[\text{TF}\]

(See, you'll check that \(T P_m = m (m+1) f.\))

In the 2nd example of Lecture 16, we showed that \(T\) is self-adjoint in the inner product \(\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) \, dx\), with the consequence that eigenvalues of \(T\) with distinct eigenvalues are orthogonal in this inner product. In particular, the \(\{P_m(x)\}\) are.