So far, we have learned a great deal about solving equations of the form \( Lf = 0 \) if \( L \) has constant coefficients, and solving \( LF = R \) in general once one has a basis of solutions to \( Lf = 0 \). But what about solving \( Lf = 0 \) if \( L \) doesn't have constant coefficients?

The problem is that the solutions may not be expressible in terms of familiar functions. (In fact one frequently uses DEs to define new transcendental functions, like the Bessel, hypergeometric, Legendre, & Airy functions.)

One natural idea is to try to solve the DE in terms of power series, and provided the coefficient functions of \( L \) are analytic — i.e. expressible in terms of (convergent) power series — this works well.

So let \( I = (x_0 - a, x_0 + a) \), and say that \( f \in A(I) \iff f(x) = \sum a_n (x - x_0)^n \) on \( I \).

We'll assume \( L = D^2 + P_1(x)D + P_2(x) \), with \( P_1, P_2 \in A(I) \); the only reason to restrict to operators
of order 2 is to keep the notation from getting too messy.

**Theorem:** The solution space \( \ker (L : A(I) \to A(I)) \) has dimension 2. (More generally, if \( L \) has order \( n \), the dimension is \( n \), though I won't prove that.)

**Proof:** Wolog \( k_0 = 0 \). Write \( P_1 = \sum b_n x^n, P_2 = \sum_{n \geq 0} c_n x^n \), \( f = \sum_{n \geq 0} a_n x^n \). We must solve \( Lf = 0 \) for \( \{a_n\} \).

- \( P_2 f = \sum_{n \geq 0} \left( \sum_{k=0}^{n} a_k c_{n-k} \right) x^n \)
- \( P_1 f' = P_1 \sum_{n \geq 0} n a_n x^{n-1} = P_1 \sum_{n \geq 0} (n+1) a_{n+1} x^n = \sum_{n \geq 0} \left( \sum_{k=0}^{n} (n+1) a_k c_{n-k} b_{n-k} \right) x^n \)
- \( f'' = \sum_{n \geq 2} n (n-1) a_n x^{n-2} = \sum_{n \geq 2} (n+2) (n+1) a_{n+2} x^n \)

\[ \Rightarrow Lf = \sum_{n \geq 0} \left\{ (n+2)(n+1) a_{n+2} + \sum_{k=0}^{n} a_k c_{n-k} + \sum_{k=0}^{n} k a_k b_{n-k+1} \right\} x^n \]

\[ \Rightarrow a_{n+2} = -\frac{\sum_{k=0}^{n} a_k c_{n-k} + \sum_{k=0}^{n} k a_k b_{n-k+1}}{(n+1)(n+2)} \]

Define \( a_{n+2} \) in terms of \( a_0, a_1, a_2, \ldots, a_n \).

\[ \Rightarrow \{a_n\} \text{ is determined by } a_0, a_1 \text{ which may be chosen freely} \]

\[ \Rightarrow \text{yield 2 independent solutions } \begin{cases} u_1 = 1 + O(x) + \text{higher-order terms} \\ u_2 = 0 + 1x + \ldots \end{cases} \]

provided the resulting power series converge on \( I \).
To check that $\sum \alpha_k x^k$ converges for $x \in \mathcal{S} = (-r, r)$, pick any $t \in (0, r)$. The radius of convergence of $P_1$ & $P_2$ are $2r$, so $|b_k| \leq \frac{M}{e^{k+1}}$, $|c_k| \leq \frac{M}{e^{k+1}}$ for some fixed $M$.

$$\Rightarrow \quad (\alpha_k)(\alpha_{k+1}) |a_{k+1}| \leq \sum_{k=0}^{\infty} |a_k| |b_k| + \sum_{k=0}^{\infty} |a_k| |c_{k+1}|
\leq \sum_{k=0}^{\infty} |a_k| \frac{M}{e^{k+1}} + \sum_{k=0}^{\infty} |a_k| \frac{M}{e^{k+1}}
\leq (\sum_{k=0}^{\infty} (k+1) |a_k| t^k) \frac{M}{t^{n+1}}.
$$

Let $A_0 = \{|a_0|\}$, $A_1 = \{|a_1|\}$, & define

$$A_{n+2} = \frac{M}{(n+1)(n+2) t^{n+1}} \sum_{k=0}^{n+1} (k+1)|A_k| t^k.$$

Then $|a_n| \leq A_n \left(\frac{M}{t^{n+1}}\right)$.

$$\Rightarrow \quad \sum a_n t^n \text{ converges if } \sum A_n t^n \text{ does.}
$$

But $(n+1)(n+2) A_{n+2} = \frac{M}{e^{n+1}} \sum_{k=0}^{n+1} (k+1) A_k t^k$

$$- \left[ \frac{n(n+1) A_{n+1}}{t} \right] = \frac{M}{e^{n+1}} \sum_{k=0}^{n+1} (k+1) A_k t^k
$$

gives

$$(n+1)(n+2) A_{n+2} = \frac{n(n+1) A_{n+1}}{t} + \frac{M}{e^{n+1}} \frac{(n+2) A_{n+1} t^n}{(n+1)(n+2) t}
$$

$$\Rightarrow \quad A_{n+2} = \frac{n(n+1) + M(n+2) t}{(n+1)(n+2) t} A_{n+1}.
$$

So

$$\lim_{n \to \infty} \frac{A_{n+2} |x|^{n+2}}{A_n |x|^n} = \lim_{n \to \infty} \frac{n(n+1) + M(n+2) t}{(n+1)(n+2) t} |x| = \frac{|x|}{t}
$$

is less than 1 (and $\sum |a_k| t^n$ converges) for $|x| < t$. Since $x \in (0, r)$ was arbitrary, $\sum |a_k| t^n$ converges for $|x| < r$. \qed
An example

\[ L = D^2 - \left( \frac{2x}{1-x^2} \right) D + \alpha(\alpha+1) \frac{1}{1-x^2}, \quad \alpha \in \mathbb{R} \]

Solutions to \( Lf = 0 \) on some \( \Omega \) then \( \hat{L} f = 0 \) when

\[ \hat{L} = (1-x^2) D^2 - 2x D + \alpha(\alpha+1). \]

They are called Legendre functions. Writing \( f = \sum a_n x^n \),

\[ 0 = (1-x^2) D^2 \sum a_n x^n - 2x D \sum a_n x^n + \alpha(\alpha+1) \sum a_n x^n \]

\[ = \sum \left\{ \frac{(n+1)(n+2) a_{n+2} - n(n-1) a_n - 2n a_n + \alpha(\alpha+1) a_n}{n} \right\} x^n \]

\[ \Rightarrow a_{n+2} = \frac{(n-a)(n+a+1)}{(n+1)(n+2)} a_n. \text{ So if you start with} \]

* \( a_0 = 1, \ a_1 = 0 \), you get an even function \( u_1 \) — and if \( \alpha \) is an even integer, \( a_n = 0 \) for \( n > \alpha \) \( \Rightarrow \) \( u_1 \) is a polynomial.

* \( a_0 = 0, \ a_1 = 1 \), you get an odd function \( u_2 \) — and if \( \alpha \) is an odd integer, \( a_n = 0 \) for \( n > \alpha \) \( \Rightarrow \) \( u_2 \) is a polynomial.

These polynomial solutions for \( \alpha = m \in \mathbb{N} \) are the Legendre polynomials

\[ P_m(x) = \sum_{r=0}^{m} \frac{(-1)^r (2m-2r)!}{2^m r! (m-r)! (m-2r)!} x^{m-2r} = \frac{1}{2^m m!} \left( \frac{d}{dx} \right)^m (x^2-1)^m. \]

In your HW, you'll check directly that these satisfy

\[ L P_m = 0 \] but in a slightly different form: assume

\[ Lf = 0 \Leftrightarrow \hat{L} f = 0 \Leftrightarrow (1-x^2) f' + \alpha(\alpha+1) f = 0 \]
\[ (x^2 - 1)f' = \alpha (x + 1) f. \]

So, you'll check that \( TP_m = m (m+1) P_m. \)

In the 2nd example of Lecture 16, we showed that \( T \) was self-adjoint in the inner product \( \langle f, g \rangle = \int_{-1}^{1} f(x)g(x) \, dx \), with the consequence that eigenvalues of \( T \) with distinct eigenvalues are orthogonal in this inner product. In particular, the \( \{P_m(x)\} \) are.